A TREATISE

ON

CONIC SECTIONS

AND THE

APPLICATION OF ALGEBRA TO GEOMETRY.

John

BY J. HYMERS, D.D.

FELLOW AND TUTOR OF ST JOHN'S COLLEGE CAMBRIDGE.

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CONIC SECTIONS

AND THE

APPLICATION OF ALGEBRA TO GEOMETRY.

SECTION I.

ON THE METHODS OF DETERMINING THE POSITION OF A POINT IN A PLANE.

Rectangular and oblique Co-ordinates. Polar Co-ordinates.

1. In order to determine the position of a point in a plane, some fixed point in the plane is taken for the origin of co-ordinates; and through it are drawn two fixed lines, called the co-ordinate axes, at right angles to one another.

Then if the perpendicular distance (to which the name *ordinate* is given) of a point from each of the co-ordinate axes be assigned, its position will be completely determined.

For let A (fig. 1) be the origin of co-ordinates, $X'AX$, $Y'AY$, the co-ordinate axes, P any point, and PM, PN, the perpendiculars let fall from it upon the co-ordinate axes; these perpendiculars together are called the rectangular *co-ordinates* of P, and as their values change for the different points of the plane, they are denoted by the variables x and y.

Then the point P will be determined in position, if we know the values of its two co-ordinates; that is, if we know that for that point $x = a$, $y = b$; for if along AX we measure $AN = a$, and through N draw an indefinite line parallel to AY, this line will contain all points in the plane whose distance from AY is a, or for which $x = a$, and therefore the point in question; similarly, if we measure along AY the distance $AM = b$, and through M draw an indefinite line parallel to AX, this line will contain the

point in question; therefore these two lines MP, NP, will, by their intersection in P, determine one single position for the point whose co-ordinates are $x = a$, $y = b$; which position, as we see, coincides with the angular point opposite the origin, of the rectangle constructed with the sides AN, AM, equal to the two given co-ordinates.

2. Instead of the perpendicular PM, its equal AN is commonly used to determine the position of the point P; and the two AN, NP, are called the co-ordinates of P, and are denoted by x and y; the former, for the sake of distinction, being called the *abscissa*, as being cut off from AX, and the latter, which is parallel to the other axis AY, the *ordinate*.

When the point is given, and consequently its co-ordinates known, they are usually represented by the first letters of the alphabet a, b, &c. as above; or by the accented letters x', y', or x'', y''; and the point is called the point (a, b), the point (x', y'), &c.; also the axes of the co-ordinates AX, AY, are often called the axis of x and the axis of y.

3. The determination of the point P will not however be complete, unless we take into account the signs of the quantities a, b, in the equations

$$x = a, \quad y = b,$$

in order to measure these distances, when they are positive, along the positive parts AX, AY, of the co-ordinate axes; or along the negative parts AX', AY', of the axes produced in the contrary direction, when they are negative; as is explained in Trigonometry, (Art. 20). For since the co-ordinate axes, which must be supposed to be prolonged indefinitely, form about the origin four angular compartments, there are four positions in which P might be situated at absolute distances a, b, from the co-ordinate axes; and it is only by attending to the algebraical signs, with which the values of those distances are affected, that we shall be enabled to select the true position of the point. The direction of the negative abscissæ is quite arbitrary, as is also that

of the negative ordinates; we shall however, according to the usual practice, measure the positive abscissæ from the origin towards the right, and the negative abscissæ from the origin towards the left; and the positive ordinates we shall measure upwards from the axis of x, and the negative ordinates, downwards. Hence if the point P be situated in the compartment XAY, both its co-ordinates are positive; if in the opposite compartment $X'AY'$, both are negative; and for points in the compartments $X'AY$, XAY', we must have respectively

$$x = -a, \quad y = b; \quad x = a, \quad y = -b;$$

also for points in the axis of x, and axis of y, we shall have respectively

$$x = a, \quad y = 0; \quad x = 0, \quad y = b;$$

and for the origin, $x = 0$, $y = 0$.

4. Sometimes it is requisite to take the co-ordinate axes not at right angles, but inclined at a given angle to one another; in which case, the system of co-ordinates is called oblique.

Thus (fig. 2), if XAX', YAY', be two lines drawn through the point A, and intersecting one another at a given angle; and if from any point P in the plane XAY, PM, PN, be drawn respectively parallel to AX, AY, and meeting those axes in M and N; PM or its equal AN, and NP, are the co-ordinates of P referred to the oblique axes AX, AY.

5. To find the distance of a point from the origin in terms of its co-ordinates.

Let P be the point (fig. 1), $AN = x'$, $NP = y'$, its given co-ordinates.

Join AP, and let $AP = d$; then from the triangle ANP, right-angled at N,

$$AP^2 = AN^2 + NP^2, \text{ or } d^2 = x'^2 + y'^2,$$

$$\therefore d = \sqrt{x'^2 + y'^2}.$$

6. To find the distance between two points in terms of their co-ordinates; and the angle of inclination of the line which joins them, to the axis of x.

Let P' be a point (fig. 3) whose co-ordinates are x' and y'; and P any other point whose co-ordinates are x and y; join $P'P$, and draw $P'Q$ parallel to AX and meeting the ordinate of P in Q; then from the triangle PQP', right-angled at Q,

$$P'P^2 = P'Q^2 + PQ^2,$$

$$\text{or } d^2 = (x - x')^2 + (y - y')^2,$$

$$\therefore\ d = \sqrt{(x - x')^2 + (y - y')^2}.$$

Both in this formula, and in that of Art. 5, we take the radical with a positive sign, as the question only relates to the absolute distance of the points.

Next, let α be the angle which $P'P$ forms with $P'Q$, and which is equal to the angle at which, if produced, $P'P$ would be inclined to the positive part of the axis of x;

$$\text{then } \tan\alpha = \frac{PQ}{P'Q} = \frac{y - y'}{x - x'}.$$

It is important to observe that by the distance $P'P$ is meant the distance measured from P' to P, and not from P to P'; and by the angle which $P'P$ forms with Ax, is meant the angle which a line AX parallel to $P'P$ through the origin would form with the axis of x, X being always on the same side of A that P is of P'.

7. Suppose the co-ordinates to be oblique, and the axes of the co-ordinates to be inclined to one another at an angle ω; then, for the distance of a point from the origin, by Trigonometry (Art. 92) we have (fig. 2)

$$AP^2 = AN^2 + NP^2 - 2AN \,.\, NP\cos ANP,$$

$$\text{but } \cos ANP = -\cos XAY = -\cos\omega,$$

$$\therefore\ d^2 = x^2 + y^2 + 2xy\cos\omega;$$

and for the distance between two points we have, in a similar manner, from the triangle PQP' (fig. 3), in which $\angle\, PQP' = \pi - P'QN = \pi - P'N'X = \pi - \omega$,

$$d^2 = (x - x')^2 + (y - y')^2 + 2(x - x')(y - y')\cos\omega.$$

8. There is also another mode of determining the position of a point in a plane, viz. by means of its distance from a given point or pole, and the angle which that distance makes with a fixed line or axis in the plane.

Let A (fig. 4), be the origin or pole, and AX a fixed line or axis; and P any point in a plane passing through AX. Join AP, then AP is called the *radius vector*, and is usually denoted by r, and the angle PAX is called the *angle of revolution*, and is denoted by θ; and r and θ are called the polar co-ordinates of P; and if given values $r = d$, $\theta = a$, be assigned for them, the position of P will be completely determined.

The angle of revolution may receive any positive value from zero to infinity; and it is measured from the initial line or axis always in the same direction, which, according to the usual practice, we shall assume to be upwards; and the radius vector is measured from the pole along the line bounding that angle, and may have any positive value from zero to infinity. Sometimes, however, in order to embrace all the branches of a curve in the same polar equation, it is necessary to admit negative values of r, and to measure them from the pole along the radius vector produced backwards; also, if negative values of θ be admitted, they must be measured from the initial line downwards.

9. To express the distance of two points from one another in terms of their polar co-ordinates.

Let P' be a point (fig. 4) whose polar co-ordinates are r' and θ', and P any other point whose co-ordinates are r and θ; then $\angle PAP' = \theta - \theta'$; and, joining PP', we get from the triangle PAP',

$$PP' \text{ or } d = \sqrt{r^2 + r'^2 - 2rr' \cos(\theta - \theta')}.$$

Equation to a Curve. Locus of an Equation.

10. As we are able, in the mode explained above, to determine the position of a point in a plane by means of its co-ordinates, we may suppose a curved line to be traced on a

plane, and each of its points to be referred to two known axes; and that we have between the abscissa and ordinate of each point an invariable relation. In a great many cases, it happens that this relation is of a nature to be expressed by an equation between the abscissa and ordinate; and this equation, when obtained, enables us to find either of those quantities by means of the other; so that, giving to the abscissa, for instance, arbitrary values, we can deduce from the equation corresponding values of the ordinate; and we thus determine as many points of the curve as we please.

The equation which expresses, generally, the invariable relation of the abscissa and ordinate of every point of a curve to one another, is called the equation to the curve: and, conversely, the curve is called the locus of the equation.

Similarly, the equation which expresses the invariable relation of the radius vector and angle of revolution of every point of a curve to one another, is called the polar equation to the curve.

11. All lines are regular, or irregular; irregular lines, described, as it is termed, *liberâ manu*, are not subjects of mathematical investigation, and cannot be represented by equations; but regular lines, described according to some constant law which determines the position of all their points, can be represented by equations. This idea of regular lines agrees with the geometrical loci of the ancients. They gave that name to those lines of which every point was equally proper to solve an indeterminate geometrical problem. Thus a circle was said to be the locus of the vertices of all triangles on a given base and having a given vertical angle. Des Cartes first adopted the method of expressing, by an algebraical equation, the nature of lines. The object of the following Sections will be to investigate the equations to curves, and from those equations to discover their geometrical properties by means of interpretations made according to the laws of Algebra.

SECTION II.

ON THE STRAIGHT LINE.

Straight Line referred to Rectangular Co-ordinates.

12. WE will now suppose the locus of the point P to be a straight line, as defined in Geometry; and proceed from some of the fundamental properties of a straight line to deduce its equation; that is, an equation expressing an invariable relation which is satisfied by the co-ordinates of every point in it.

13. To find the equation to a straight line.

Let A be the origin (fig. 5), AX the axis of x, AY that of y; RT the given straight line meeting these axes in T and B respectively, P any point in it, and $AN = x$, $PN = y$ the co-ordinates of P.

Let $AB = c$, and the tangent of angle $PTN = m$. Draw BQ parallel to AX, meeting PN in Q; then by Trigonometry (Art. 90)

$$PQ = BQ . \tan PBQ = AN . \tan PTN = mx,$$

$$\text{and } PN = PQ + QN = PQ + AB,$$

$$\therefore y = mx + c;$$

and as this relation is satisfied by the co-ordinates of every point in the line, it is the equation required.

OBS. The meanings of the constants m and c are to be particularly noticed; c is the part of the axis of y intercepted between the straight line and the origin, or the ordinate through the origin; m is the tangent of the angle which that part of the line which falls above the axis of x,

makes with the axis of x produced in the positive direction. They remain the same for the same line, but are different for different lines, and are called arbitrary constants, or parameters; in general every straight line has two of them, and therefore a straight line may be drawn fulfilling two conditions.

14. The equation $y = mx + c$, which is the most convenient form and the one commonly employed, represents a straight line when determined by the conditions of passing through a known point in the axis of y, and making a given angle with a fixed line, viz. the axis of x; so that m is a number or ratio, denoting the tangent of the angle; and c denotes a line, viz. the distance from the origin, of the point in the axis of y through which the line passes.

If $c = 0$, the line passes through the origin, and its equation is $y = mx$; also if $m = 0$, the line is parallel to the axis of x, and its equation is $y = c$. Similarly, the equation $x = a$, since it belongs to all points whose distance from the axis of y is a, represents an indefinite line parallel to that axis; and the equations $y = 0$, $x = 0$, represent the axis of x, and the axis of y, repectively, (Arts. 1 and 3).

15. The equation to a straight line may also be put under the two following forms, which are sometimes useful.

Let BT (fig. 7) be the line intersecting the positive parts of both the co-ordinate axes, and let $AB = c$, $\tan BTX = m$, as before; and $AT = a$; then the equation to the line is

$$y = mx + c;$$

$$\text{but } m = \tan BTX = -\tan BTA = -\frac{c}{a},$$

$$\therefore y = -\frac{c}{a}x + c, \quad \text{or } \frac{x}{a} + \frac{y}{c} = 1,$$

the equation to a straight line when determined by means of the portions of the positive co-ordinate axes intercepted between it and the origin.

Also, if a perpendicular upon the line from the origin, $AD = p$, and $\angle DAX = a$, we have, by Trig. (Art. 23),

$$m = -\tan BTA = -\cot a, \text{ and } c = \frac{p}{\sin a} \text{ (Trig. 98)},$$

$$\therefore y = -x \cot a + \frac{p}{\sin a}; \text{ or } y \sin a + x \cos a = p,$$

the equation to a straight line when determined by the perpendicular upon it from the origin, and the angle which the perpendicular makes with the axis of x produced in the positive direction.

16. The indeterminate equation of the first degree between two variables, is, in its most general form,

$$Ax + By + C = 0,$$

which in all cases is the equation to a straight line. For by putting

$$\frac{A}{B} = -m, \quad \frac{C}{B} = -c,$$

we reduce it to the form $y - mx - c = 0$, or $y = mx + c$, which coincides with the equation to a straight line.

17. A straight line may always be constructed from its equation $y = mx + c$, when the constants m and c are known.

First, consider the equation $y = mx$, which represents a line passing through the origin; assume for x any positive value $AN = x'$ (fig. 6), take for y the value $NP = mx'$ (supposing m a positive quantity) and join AP, this produced indefinitely both ways is the required line. But if m be negative, so that the equation is $y = -mx$, taking $AN = x'$, and NP measured downwards $= mx'$ (fig. 7), and joining AP, we have the line required.

We can now readily construct any line whatever whose equation is $y = mx + c$.

For if in this equation, we give to x the same values that we assigned to it in the equation $y = mx$, the difference of the corresponding values of y will be constant and equal to c; the straight line which is the locus of $y = mx + c$, is consequently parallel to the line AP (fig. 6) determined by $y = mx$ (14); if therefore we take AB equal to c, and draw BD parallel to AP, we shall have the line required; for it will be such that for every point Q in it, $PQ = AB$. If c be negative, we must take AB' equal to c and draw $B'D'$ parallel to AP; and if m be negative, so that the proposed equation is $y = -mx + c$, then we must take AB or AB' (fig. 7) equal to c, and draw BD or $B'D'$ parallel to AP.

18. As a straight line is determined when any two points are known through which it passes, the position of the line which is the locus of any indeterminate equation of the first degree, may also be assigned by determining two of its points; and for this purpose the points most convenient are those in which it cuts the axes of y and x; the co-ordinates of which are obtained by making x and y successively equal to zero in the given equation.

Thus if the equation be $y = mx + c$, in which m and c are positive, making $x = 0$, we have $y = AB = c$ (fig. 5), and making $y = 0$, we have $-x = AT = \frac{c}{m}$, (3). Hence, joining TB and producing it indefinitely, we have the line required.

The distances of the points of intersection from the origin, determined in this manner, must of course be measured along the positive or negative parts of the co-ordinate axes, accordingly as they are affected with positive or negative signs.

Problems relative to the Straight Line.

19. These principles being laid down, we proceed to the resolution of several problems relative to the straight line, the results of which are of great use. As the equation to a straight line contains two disposable constants (13),

they may be determined so as to make the line fulfil various conditions; as, for instance, to pass through two given points; to pass through a given point and be parallel, or perpendicular, or inclined at a known angle, to a given line; and so on.

To find the equation to a straight line which shall pass through the origin, and through a given point.

Let x', y', be the co-ordinates of the given point P (fig. 6); then the equation to the line will be (Art. 14) $y = mx$, where m is to be determined; but

$$m = \tan PAN = \frac{PN}{AN} = \frac{y'}{x'};$$

therefore the required equation is $y = \dfrac{y'}{x'}\, x$.

20. To find the equation to a straight line which shall pass through a given point, and make a given angle with the axis of x.

Let x', y', be the co-ordinates of the given point, and m the tangent of the given angle; then the equation to the line will be (Art. 14) $y = mx + c$, where c is to be determined. But since x', y', are the co-ordinates of a point in the line, they will satisfy its equation; $\therefore\ y' = mx' + c$, which gives $c = y' - mx'$; and substituting this value of c, we get for the required equation,

$$y - y' = m(x - x').$$

21. To find the equation to a straight line which shall pass through two given points, whose co-ordinates are x', y'; x'', y''.

Any point in the line, of which the co-ordinates are x and y, being assumed, we have $y = mx + c$. But since x' and y' are also co-ordinates of a point in the same line, they will satisfy this equation,

$$\therefore\ y' = mx' + c.$$

Hence, subtracting this equation from the former, we have

$$y - y' = m(x - x').$$

This is the equation to a line fulfilling one condition, viz. passing through the point (x', y'); and since m is arbitrary, an infinite number of lines may be so drawn.

But since the line is moreover to pass through the point (x'', y''), its equation will be satisfied by putting $x = x''$, $y = y''$,

$$\therefore\ y'' - y' = m(x'' - x'), \quad \text{or } m = \frac{y'' - y'}{x'' - x'};$$

so that m is no longer arbitrary, but expressed in terms of the given quantities; hence, substituting this value of m in the above equation, we get

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x'),$$

the equation required; which may be also written

$$y = \frac{y'' - y'}{x'' - x'}\,x + \frac{y'x'' - x'y''}{x'' - x'},$$

and is then of the general form $y = mx + c$.

22. To find the equation to a straight line which shall pass through a given point and be parallel to a given line.

Let the equation to the given line be $y = mx + c$, where m and c are known; and the equation to the required line

$$y = m'x + c',$$

where m' and c' are unknown; then, in order that these lines may be parallel, we must have $m = m'$; for they are respectively parallel to lines passing through the origin whose equations are

$$y = mx, \quad y = m'x,$$

and these two latter lines must coincide, since the two former are parallel; consequently $m = m'$.

The equation to the line parallel to the given line then becomes

$$y = mx + c',$$

where c' remains indeterminate, since there is an infinite number of lines which are parallel to a given line; but if the parallel be required to pass through a given point (x', y'), we must have

$$y' = mx' + c', \quad \text{which gives } c'.$$

Subtracting therefore this from the preceding, c' will disappear; and we have, for the equation to the required parallel through a given point,

$$y - y' = m(x - x').$$

23. To find the equation to a straight line which shall pass through a given point, and be perpendicular to a given line.

Let x', y' be the co-ordinates of the given point, and $y = mx + c$, the equation to the given line; then if AD (fig. 7) be drawn through the origin parallel to the given line, its equation will be $y = mx$; let BT be the required line, then its equation will be

$$y - y' = m'(x - x'),$$

$$\text{but } m' = \tan BTX = -\tan BTA = -\cot DAT = -\frac{1}{m},$$

$\therefore$ the required equation is

$$y - y' = -\frac{1}{m}(x - x').$$

Hence if the coefficient of x in one equation, be the reciprocal of the coefficient of x in the other with a contrary sign, that is, if the equations be $y = mx + c$, $y = -\frac{1}{m}x + c'$, the lines which they represent are at right angles to one another.

24. Having given the equations to two straight lines, to determine the co-ordinates of their point of intersection.

Let the equations to the two lines be

$$y = mx + c,$$
$$y = m'x + c'.$$

At the point where these lines intersect, they have the same co-ordinates; and, conversely, their co-ordinates are not equal at any other point, except that in which they intersect; hence for that point only we have

$$mx + c = m'x + c', \text{ which gives } x = \frac{c - c'}{m' - m};$$

and substituting this value for x in the equation to either of the lines, we get

$$y = m \cdot \frac{c - c'}{m' - m} + c, \text{ or } y = \frac{m'c - mc'}{m' - m}.$$

When $m' = m$, these values become infinite, as ought to happen, for the lines are then parallel (22); when $c' = c$ as well as $m' = m$, the values become $\frac{0}{0}$, that is to say, indeterminate, which likewise ought to happen, as the lines then become coincident in all their points. If the lines are perpendicular to one another, so that $m' = -\frac{1}{m}$, the co-ordinates of their point of intersection become

$$x = \frac{m(c' - c)}{1 + m^2}, \quad y = \frac{c + m^2c'}{1 + m^2}.$$

Hence the equation to any line passing through the point of intersection of the proposed lines, will be

$$y - mx - c = n(y - m'x - c'), \text{ where } n \text{ is arbitrary.}$$

25. Having given the equations to two straight lines, to find the angle between them.

$$\text{Let } y = mx + c,$$
$$y = m'x + c',$$

be the equations to the lines BT, $B'T'$, (fig. 8); and α, α', the angles at which they are respectively inclined to the axis

of x, so that (13) $\tan \alpha = m$, $\tan \alpha' = m'$; and let ϕ be the angle between them,

Then $\angle BPB' = \angle BTX - \angle B'T'X$, or $\phi = \alpha - \alpha'$;

$$\therefore \text{ by Trig. Art. 45, } \tan \phi = \frac{\tan \alpha - \tan \alpha'}{1 + \tan \alpha . \tan \alpha'},$$

$$\text{or } \tan \phi = \frac{m - m'}{1 + mm'}.$$

Similarly, $\sin \phi = \sin \alpha . \cos \alpha' - \cos \alpha . \sin \alpha' = \dfrac{m - m'}{\sqrt{1 + m^2} . \sqrt{1 + m'^2}}$,

$$\cos \phi = \frac{1 + mm'}{\sqrt{1 + m^2} . \sqrt{1 + m'^2}}. \text{ (Trig. Art. 19).}$$

Hence, in order that the two lines may be parallel, we must have $\tan \phi = 0$, or $m - m' = 0$, as before, (22).

And in order that they may be at right angles to one another, we must have $\tan \phi = \infty$, or $1 + mm' = 0$.

26. To find the equations to the straight lines which pass through a given point, and make a given angle with a given straight line.

Let BT (fig. 8) be the given line, and $y = mx + c$ its equation, therefore $\tan PTA = m$; $B'T'$ one of the required lines whose equation may be assumed to be $y - y' = m'(x - x')$ since it passes through the point (x', y'), (20) then $\tan PT'A = m'$; also let $\tan TPT' = t$, a given quantity; then because

$$\angle PT'A = PTA - TPT', \text{ or } \alpha' = \alpha - \phi,$$

$$\tan \alpha' = \frac{\tan \alpha - \tan \phi}{1 + \tan \alpha . \tan \phi}, \text{ or } m' = \frac{m - t}{1 + mt};$$

therefore $y - y' = \dfrac{m - t}{1 + mt}(x - x')$ is the required equation.

Similarly if PT'' be the other line answering the conditions of the Problem and $\angle PT''X = \alpha''$ then $\alpha'' = \alpha + \phi$,

$\therefore m'' = \dfrac{m + t}{1 - mt}$ and the required equation is $y - y' = \dfrac{m + t}{1 - mt}(x - x')$.

If the lines are to be parallel, $t = 0$, and as before the equation is

$$y - y' = m(x - x').$$

If the lines are to be perpendicular to one another, $t = \infty$, and therefore the equation (since m in the numerator, and 1 in the denominator, vanish with respect to t) becomes

$$y - y' = \frac{-t}{mt}(x - x'), \text{ or } y - y' = -\frac{1}{m}(x - x').$$

27. Having given the co-ordinates of a point, and the equation to a straight line; to find the length of the perpendicular dropped from the point upon the line.

Let x', y', be the co-ordinates of the given point, and $y = mx + c$ the equation to the given line; then the equation to the perpendicular will be

$$y - y' = -\frac{1}{m}(x - x').$$

In order to get the co-ordinates of the point of intersection of the given line and the perpendicular, we must, as in Art. 24, deduce from their equations values of x and y; to make the process easier, put the first, $y = mx + c$, under the form

$$y - y' = m(x - x') + c + mx' - y';$$

combining this with the equation to the perpendicular, and taking for the unknown quantities, the differences $y - y'$, $x - x'$, we get

$$x - x' = \frac{m(y' - mx' - c)}{1 + m^2}, \quad y - y' = -\frac{y' - mx' - c}{1 + m^2};$$

values from which it is easy to deduce the co-ordinates x, and y, of the foot of the perpendicular. But if we denote by p the length of the perpendicular intercepted between the point and the given line, we have (Art. 6),

$$p = \sqrt{(x - x')^2 + (y - y')^2};$$

therefore, putting for $x - x'$ and $y - y'$ their values,

$$p = \pm \frac{y' - mx' - c}{\sqrt{1 + m^2}}.$$

As the value of p must be positive, we must take the upper or lower sign, according as the numerator $y' - mx' - c$ is positive or negative.

The double sign may be explained, as having reference to the face of the straight line upon which the perpendicular falls. For suppose the line to revolve about B from the present position, in which the perpendicular is positive; then as it moves up to P, the perpendicular diminishes, and vanishes; and when it passes P, the perpendicular, falling upon a different face, becomes negative, and continues so till the line, after half a revolution, returns to its first position; the line has then the same equation as before, but has a different face turned towards P; and the perpendicular has the same value as at first, but with a contrary sign (fig. 9).

If the given point is situated in the origin, $x' = 0$, $y' = 0$, and the value of p is reduced to

$$p = \pm \frac{c}{\sqrt{1 + m^2}}.$$

28. The result of the preceding Article may be readily obtained as follows.

Let $y = mx + c$ be the equation to the given line MT (fig. 9), and $AN = x'$, $PN = y'$, the co-ordinates of the point P; then the perpendicular

$$PQ = PR \cos RPQ = PR \cos RTN.$$

But $PN = y'$, and $RN = mx' + c$, $\therefore PR = y' - mx' - c$;

$$\text{also } \cos RTN = \frac{1}{\sqrt{1 + \tan^2 RTN}} = \frac{1}{\sqrt{1 + m^2}};$$

$$\therefore \text{ the perpendicular } p = \frac{y' - mx' - c}{\sqrt{1 + m^2}}.$$

29. Hence also, if through a given point a line be drawn cutting a given line at a known angle, we can find the distance of the given point from the point of intersection of the lines. For if PS (fig. 9) be a line passing through the point P, and cutting the line BM at an angle $PSM = a$, by drawing PQ perpendicular to BM, we have

$$SP \sin a = QP = \frac{y' - mx' - c}{\sqrt{1 + m^2}}; \quad \therefore SP = \frac{y' - mx' - c}{\sin a \sqrt{1 + m^2}}.$$

Straight Line referred to Oblique Co-ordinates.

30. To find the equation to a straight line referred to oblique co-ordinates.

Let the axes of the co-ordinates be inclined to one another at an angle ω, and suppose the line PT (fig. 10), to cut the axis of x at an angle a. Let $AN = x$, $NP = y$, be co-ordinates of any point P, and draw BQ parallel to AX, meeting the ordinate of P in Q;

$$\text{then } \frac{PQ}{BQ} = \frac{\sin PBQ}{\sin BPQ} = \frac{\sin a}{\sin (\omega - a)}, \text{ or } PQ = \frac{x \sin a}{\sin (\omega - a)},$$

and $NQ = AB = c$; therefore the required equation is

$$y = \frac{x \sin a}{\sin (\omega - a)} + c.$$

Hence if a straight line referred to oblique axes, be represented by the equation $y = mx + c$; m, the coefficient of x, expresses the ratio of the sines of the angles which the line makes respectively with the axes of x and y; and c, as before, is the ordinate through the origin.

In using the equation $y = \dfrac{x \sin a}{\sin (\omega - a)} + c$, we must remember, with respect to the constants involved, (1) that c is a positive or negative quantity, according as the line cuts the axis of y above or below the origin; (2) that ω is the angle YAX formed by the positive parts of the

co-ordinate axes, and not the adjacent angle YAX'; and (3) that α is the angle PTX formed by the portion of the line which is situated above the axis of x, with the positive part of that axis.

31. It is easily seen that when a line is determined by the portions of the co-ordinate axes intercepted between it and the origin, its equation is of precisely the same form as when the co-ordinates are rectangular; for let PT be the line (fig. 12), $AP = c$, $AT = a$; and let x and y be the co-ordinates of any point Q; we get from the similar triangles APT, QNT,

$$\frac{AP}{AT} = \frac{QN}{NT}, \quad \text{or } \frac{c}{a} = \frac{y}{a-x}, \quad \text{or } \frac{x}{a} + \frac{y}{c} = 1.$$

Also when a line is determined by the condition of passing through two given points, or of being parallel to a given line, its equation is of the same form whether the co-ordinates be rectangular or oblique; in the following cases, the results are different.

32. To find the angle between two lines whose equations are given, referred to oblique axes.

First, to calculate the angle which a line whose equation is given, makes with the axis of x. Let $y = mx + c$ be its equation, and ω the angle of inclination of the axes; and let the line make an angle α with the axis of x, and therefore an angle $\omega - \alpha$ with the axis of y, then

$$\frac{\sin\alpha}{\sin(\omega - \alpha)} = m,$$

$$\text{which gives } \tan\alpha = \frac{m\sin\omega}{1 + m\cos\omega}.$$

Next, let $y = m'x + c'$ be the equation to another line referred to the same axes, making an angle α' with the axis of x;

$$\therefore \tan\alpha' = \frac{m'\sin\omega}{1 + m'\cos\omega}.$$

Let ϕ be the angle between the lines,

$$\text{then } \tan\phi = \tan(\alpha' - \alpha) = \frac{\tan\alpha' - \tan\alpha}{1 + \tan\alpha' . \tan\alpha}$$

$$= \frac{(m' - m)\sin\omega}{1 + mm' + (m + m')\cos\omega}.$$

33. Hence if $\phi = \frac{1}{2}\pi$, then

$$1 + mm' + (m + m')\cos\omega = 0;$$

$$\therefore m' = -\frac{1 + m\cos\omega}{m + \cos\omega},$$

the condition in order that two lines referred to oblique axes, may be perpendicular to one another. The condition of parallelism is the same as for rectangular co-ordinates.

34. To find the perpendicular distance of a given point from a given line, referred to oblique axes.

Let x', y', be the co-ordinates of the given point Q (fig. 11), and $y = mx + c$ the equation to the given line CN which makes an angle α with the axis of x; then

$$\sin\alpha = m\sin(\omega - \alpha);$$

also let QP be the perpendicular let fall from Q upon CN; then

$$QP = QN . \sin(\omega - \alpha) = (y' - mx' - c)\frac{\sin\alpha}{m},$$

$$\text{but } \tan\alpha = \frac{m\sin\omega}{1 + m\cos\omega}, \text{ and } \therefore \sin\alpha = \frac{\tan\alpha}{\sqrt{1 + \tan^2\alpha}}$$

$$= \frac{m\sin\omega}{\sqrt{1 + 2m\cos\omega + m^2}} \text{ by substitution;}$$

$$\therefore QP = \frac{(y' - mx' - c)\sin\omega}{\sqrt{1 + 2m\cos\omega + m^2}}.$$

35. To find the area of a trapezium.

Take one of the sides of the trapezium $P'N'NP$ (fig. 3) for the axis of x, and a line parallel to its two parallel sides

for that of y; and let x, y be co-ordinates of P, and x', y' those of P'.

Then area of $P'N'NP$ = area of parallelogram $P'N$
+ area of triangle $P'PQ$

$$= (x - x')\, y' \sin\omega + \tfrac{1}{2}(x - x')(y - y') \sin\omega$$
$$= \tfrac{1}{2}(x - x')(y + y') \sin\omega.$$

Straight Line referred to Polar Co-ordinates.

36. To find the polar equation to a straight line.

Let A be the pole (fig. 12), XA the initial line, PT the proposed straight line, and $AP = r$, $\angle XAP = \theta$ the polar co-ordinates of any point P in it; $AQ = p$ a perpendicular upon it from the pole, and $\angle XAQ = \alpha$ the angle which that perpendicular makes with the initial line;

$$\text{then } AP = AQ \sec QAP,$$
$$\text{or} \qquad r = p \sec(\theta - \alpha).$$

If the proposed line be perpendicular to the initial line so that $\alpha = 0$, the equation becomes $r = p \sec\theta$. And if we choose to determine the line by the angle, and by the distance from the pole, at which it cuts the initial line, so that $AT = a$, $\angle QTX = \beta$, then $\dfrac{r}{a} = \dfrac{\sin\beta}{\sin(\beta - \theta)}$.

37. Since the polar equation to a straight line becomes

$$r\cos(\theta - \alpha) = p, \quad \text{or} \quad r\cos\theta \,.\, \cos\alpha + r\sin\theta \,.\, \sin\alpha = p;$$

it appears that every equation of the form

$$Ar\cos\theta + Br\sin\theta + C = 0,$$

is the polar equation to a straight line.

SECTION III.

ON THE TRANSFORMATION OF CO-ORDINATES.

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38. As the equation to a curve does not remain the same when we change the axes of the co-ordinates to which it is referred; it is of great importance, in investigating the form and properties of a curve from its equation, to give the origin of the co-ordinates such a position, and the axes such directions and inclination, as will allow the equation to the curve to appear under the simplest form possible.

Moreover it is frequently required, when we know the equation to a curve referred to an assumed system of axes, to find the equation which represents the same curve when referred to new axes, whose positions are given with respect to the former.

This will be effected, when we know, for any point of the curve, the values of the old co-ordinates in terms of the new ones; for then, by substituting these values in the proposed equation, we shall obtain a relation between the new co-ordinates, which is true for every point of the curve under consideration. In this consists the transformation of co-ordinates. The problem therefore to be resolved is, to express the primitive co-ordinates of any point in terms of the new co-ordinates and of the known quantities which fix the position of the new origin and axes; which we shall now consider in the following separate cases.

It is evident that if in any equation we write $-x$ for x, the effect will be to reverse the positive direction of the abscissæ; and similarly, for the ordinates.

39. To change the origin of co-ordinates without altering the direction of the axes.

Let A (fig. 15) be the origin of co-ordinates, and $AN = x$, $PN = y$, the co-ordinates of any point P; $AM = h$, $MA' = k$,

the co-ordinates of the new origin A', and $A'N' = x'$, $N'P = y'$, the co-ordinates of the same point P referred to the new axes $A'X'$, $A'Y'$, parallel to the former; then

$$x = AM + A'N' = h + x', \text{ and } y = A'M + N'P = k + y',$$

which are values of the primitive co-ordinates in terms of the new co-ordinates; and if these values be substituted for x and y, we shall have the equation to the curve referred to the origin A', the axes retaining their former directions.

40. To change the directions of the axes without altering the origin, supposing both systems to be rectangular.

Let $XAX' = YAY' = \theta$ (fig. 16) be the angle at which the new axes AX', AY', are inclined to the original axes AX, AY.

Let $AN = x$, $PN = y$; $AN' = x'$, $PN' = y'$; be the co-ordinates of the same point P referred to these respective axes.

Draw $N'Q$, $N'M$, parallel and perpendicular to AX,

$$\text{then } x = AM - QN' = x' \cos\theta - y' \sin\theta,$$
$$y = MN' + PQ = x' \sin\theta + y' \cos\theta,$$

which are values of the primitive co-ordinates in terms of the new co-ordinates; and if these values be substituted for x and y, we shall have the equation to the curve referred to the axes AX', AY'.

41. To change the direction of the axes without altering the origin, supposing both systems to be oblique.

Let AX, AY, (fig. 17) be the primitive axes inclined to one another at an angle $XAY = \omega$, and AX', AY', the new axes determined by the angles $X'AX = \alpha$, $Y'AX = \beta$, which they respectively form with the primitive axis of x.

Let P be any point, $AN = x$, $NP = y$, its primitive co-ordinates, and $AN' = x'$, $N'P = y'$, its new co-ordinates.

Then, denoting by xy the angle XAY contained by the axes of x and y produced in the positive directions, and similarly of the others, and drawing PQ, PR, perpendicular to AX, AY, and $N'S$, $N'T$, parallel to those lines, we have

$$RP = RT + TP,$$

$$\text{or } x \sin xy = x' \sin x'y + y' \sin y'y;$$

$$\therefore\ x = x' \frac{\sin(\omega - \alpha)}{\sin \omega} + y' \frac{\sin(\omega - \beta)}{\sin \omega}.$$

$$\text{Also}\quad QP = QS + SP,$$

$$\text{or } y \sin xy = x' \sin x'x + y' \sin y'x;$$

$$\therefore\ y = x' \frac{\sin \alpha}{\sin \omega} + y' \frac{\sin \beta}{\sin \omega};$$

and if these values of the primitive co-ordinates, which are expressed in terms of the new co-ordinates and known quantities, be substituted for x and y in the equation to a curve, we shall obtain the equation to the curve referred to the new axes.

Obs. In making use of these formulæ, we must recollect that ω represents the angle XAY contained by those portions of the primitive axes along which the positive co-ordinates are measured, and varies from zero to π; and that α and β denote the angles formed by AX', AY', the positive directions of the new axes, with the positive part AX of the primitive axis of x; and that each of these may receive any value from zero to 2π.

42. The formulæ of the preceding Art. are not often employed in the above general state; and we may deduce from them, as particular cases, the following, the use of which is more frequent.

First, by making $\omega = \dfrac{\pi}{2}$ and $\beta = \dfrac{\pi}{2} + \alpha$, we fall upon

the formulæ of the second case for passing from one system of rectangular co-ordinates to another system also rectangular.

Secondly, by making $\omega = \frac{\pi}{2}$, we obtain the formulæ for passing from a rectangular to an oblique system of co-ordinates, which are

$$x = x' \cos \alpha + y' \cos \beta,$$

$$y = x' \sin \alpha + y' \sin \beta.$$

Thirdly, by making $\beta = \frac{\pi}{2} + \alpha$, we obtain formulæ for passing from an oblique to a rectangular system of co-ordinates, which are

$$x = \frac{x' \sin (\omega - \alpha)}{\sin \omega} - \frac{y' \cos (\omega - \alpha)}{\sin \omega},$$

$$y = \frac{x' \sin \alpha}{\sin \omega} + \frac{y' \cos \alpha}{\sin \omega}.$$

43. In all the preceding transformations except the first, we have supposed the origin to remain unaltered; if, however, the origin is to be changed as well as the direction of the axes, we must employ the formulæ

$$x = x'' + h, \quad y = y'' + k,$$

where h, k are the co-ordinates of the new origin parallel to the primitive axes, and x'', y'', denote the values of x and y, found in each of the preceding cases.

44. To transform rectangular into polar co-ordinates, and conversely.

Let $AN = x$, $PN = y$, (fig. 4) be the rectangular co-ordinates of any point P in a curve referred to the rectangular axes AX, AY; and let $AP = r$, $\angle PAX = \theta$, be the polar co-ordinates of P.

Then $AN = AP \cos PAN$, or $x = r \cos\theta$,

$PN = AP \sin PAN$, or $y = r \sin\theta$;

and any equation between x and y may be transformed into the polar equation, by substituting these values for x and y.

Also, since $r = \sqrt{x^2 + y^2}$, and $\tan\theta = \frac{y}{x}$ or $\theta = \tan^{-1}\frac{y}{x}$, any equation between r and θ may be transformed to rectangular co-ordinates, by substituting these values of r and θ.

If the pole do not coincide with the origin of rectangular co-ordinates, then if h and k be the co-ordinates of the pole, the quantities to be substituted for x and y, in order to get the polar equation, will be

$$x = h + r\cos\theta, \quad y = k + r\sin\theta.$$

Or, if we wish at the same time to take for the initial line, not a parallel to the axis of x but a line inclined to it at an angle α, the substitutions will be

$$x = h + r\cos(\theta + \alpha), \quad y = k + r\sin(\theta + \alpha).$$

SECTION IV.

ON THE CIRCLE.

Equation to a Circle under various Forms.

45. To find the equation to a circle when referred to two diameters at right angles to one another as axes.

Let P be any point in the circumference (fig. 18), $CN = x$, $NP = y$, its co-ordinates, and $CP = c$, the radius; then from the right-angled triangle CNP,

$$CN^2 + NP^2 = CP^2,$$

$$\text{or } x^2 + y^2 = c^2,$$

which is true for every point in the circumference, and is, therefore, the required equation; it expresses that the distance of every point in the curve from the origin, is equal to c.

46. To find the equation to a circle when referred to any rectangular axes.

Let C be the center, and P any point in the circumference (fig. 19); draw CB, PN, perpendicular to AX, and CM parallel to AX, and let the co-ordinates of C be $AB = a$, $BC = b$; the co-ordinates of P, $AN = x$, $NP = y$; and the radius $CP = c$.

Then from the right-angled triangle CPM,

$$CM^2 + MP^2 = CP^2.$$

But $CM = BN = x - a$, $MP = PN - CB = y - b$;

therefore we get for the general equation to the circle referred to rectangular axes,

$$(x - a)^2 + (y - b)^2 = c^2.$$

47. From this we may deduce several particular forms of the equation to the circle, which are worthy of notice.

First, putting $a = 0$, $b = 0$, we have the origin in the centre, and fall upon the equation already found,

$$x^2 + y^2 = c^2.$$

Secondly, putting $a = c$, $b = 0$, we have the origin at the extremity of a diameter, and that diameter the axis of x, and we get

$$(x - c)^2 + y^2 = c^2,$$

or, reducing, $$y^2 = 2cx - x^2.$$

Thirdly, putting $b = 0$ simply, the centre will be in the axis of x, but the origin will not be in the circumference; similarly, putting $a = 0$, the centre will be in the axis of y; and in these two positions, the equations will be, respectively,

$$(x - a)^2 + y^2 = c^2,$$
$$x^2 + (y - b)^2 = c^2.$$

48. The above general equation to the circle (Art. 46), when developed, assumes the form

$$x^2 + y^2 - 2ax - 2by + (a^2 + b^2 - c^2) = 0,$$
$$\text{or } x^2 + y^2 + Ax + By + C = 0,$$

which does not contain the product of the variables x and y, and in which the coefficient of each of the squares of x and y, is unity. Whenever, therefore, an equation of the second order between rectangular co-ordinates, is such (or by dividing by the coefficient of x^2 can be made such) that these conditions are satisfied, the equation cannot represent any other curve except a circle. In fact, by completing the squares, we get

$$(x + \tfrac{1}{2}A)^2 + (y + \tfrac{1}{2}B)^2 = \tfrac{1}{4}A^2 + \tfrac{1}{4}B^2 - C,$$

which evidently represents a circle, the co-ordinates of whose centre are $-\frac{1}{2}A$, $-\frac{1}{2}B$; and of which the radius

$$= \sqrt{\tfrac{1}{4}A^2 + \tfrac{1}{4}B^2 - C}.$$

The equation, however, will not in reality represent a circle,

unless the quantity $\frac{1}{4}A^2 + \frac{1}{4}B^2 - C$ is positive; if this quantity is zero, the circle is reduced to a point, namely the centre; if it is negative, the equation is impossible.

Peculiarities of this sort are offered by the equations

$$x^2 + y^2 - 8y - 12x + 52 = 0,$$

$$x^2 + y^2 - 4y + \ \ 2x + \ \ 9 = 0,$$

which may be, respectively, reduced to the forms

$$(x - 6)^2 + (y - 4)^2 = 0, \quad (x + 1)^2 + (y - 2)^2 = -4.$$

But the equation $x^2 + y^2 + 4y - 4x - 8 = 0$, by completing the squares, becomes $(x - 2)^2 + (y + 2)^2 = 16$, which represents a circle whose radius is 4, and the co-ordinates of the centre 2 and -2.

49. To find the equation to a circle when referred to oblique axes.

Let, as before, $AB = a$, $BC = b$; $AN = x$, $NP = y$; (fig. 20) be the co-ordinates of the center, and of any point in the circumference of a circle, referred to the oblique axes AX, AY, which form with one another an angle ω. Draw CM parallel to AX; then from the triangle CPM in which $\angle CMP = \pi - \omega$, and whose sides are respectively equal to $AN - AB$ or $x - a$, $PN - BC$ or $y - b$, and $CP = c$ the radius, we get

$$(x - a)^2 + (y - b)^2 + 2(x - a)(y - b)\cos\omega = c^2,$$

the required equation.

50. The above equation, when developed, becomes

$$x^2 + y^2 + 2xy\cos\omega - 2(a + b\cos\omega)x - 2(b + a\cos\omega)y$$
$$+ a^2 + b^2 + 2ab\cos\omega - c^2 = 0,$$

which is of the form

$$x^2 + y^2 + 2xy\cos\omega + Ax + By + C = 0.$$

Whenever, therefore, an equation of the second order between oblique co-ordinates of known inclination, is such (or by dividing by the coefficient of x^2 can be made such) that

the two squares x^2 and y^2 have unity for coefficient, and the rectangle xy has for coefficient twice the cosine of the angle between the axes, the equation will in general represent a circle; and the co-ordinates of its centre, a and b, and the radius c, may be determined by the equations $a + b\cos\omega = -\frac{1}{2}A$, $b + a\cos\omega = -\frac{1}{2}B$, $a^2 + b^2 + 2ab\cos\omega - c^2 = C$.

Also, dropping the perpendiculars CD, CE, (fig. 20) upon the axes, we get

$$AD = AB + BD = a + b\cos\omega = -\tfrac{1}{2}A,$$

$$AE = b + a\cos\omega = -\tfrac{1}{2}B.$$

If therefore we take the distances AD, AE, equal, respectively, to half the coefficients of x and y with contrary signs, and erect the perpendiculars CD, CE, we shall by the intersection of the perpendiculars determine the position of the centre C.

Thus, in order that the equation

$$x^2 + xy + y^2 - 2ax - 2ay + a^2 = 0$$

may represent a circle, we must have $2\cos\omega = 1$, or $\omega = \frac{1}{3}\pi$; and if x', y', be the co-ordinates of its centre, and c its radius,

$$x' + y'\cos\omega = a,\ y' + x'\cos\omega = a,\ x'^2 + y'^2 + 2x'y'\cos\omega - c^2 = a^2;$$

$$\therefore\ x' = y' = \frac{2a}{3},\ \text{and}\ c = \tfrac{1}{3}a\sqrt{3}.$$

51. To find the polar equation to a circle.

Taking the pole for the origin, let a, b, be co-ordinates of the centre of the circle (fig. 21), and c its radius; and let $AP = r$, $\angle XAP = \theta$, be the polar co-ordinates of any point P, supposing the initial line to coincide with the axis of x.

Then, substituting $r\cos\theta$ for x, and $r\sin\theta$ for y, in the equation

$$(x - a)^2 + (y - b)^2 = c^2,\ \text{and expanding, we get}$$

$$r^2\cos^2\theta - 2ar\cos\theta + a^2 + r^2\sin^2\theta - 2br\sin\theta + b^2 = c^2,$$

$$\text{or}\ r^2 - 2(a\cos\theta + b\sin\theta)r + a^2 + b^2 - c^2 = 0,$$

(since $\cos^2\theta + \sin^2\theta = 1$) the required equation.

This equation will give two values of r, AP, AP';

$$\therefore\ AP \,.\, AP' = a^2 + b^2 - c^2, \text{ which is invariable.}$$

52. If we suppose the pole to be in the circumference, and the initial line to be a diameter, we have $b = 0$, $a = c$, and the equation becomes

$$r^2 - 2cr\cos\theta = 0,$$

$$\therefore\ r = 2c\cos\theta;$$

at which we may arrive immediately by joining AP, PB, (fig. 18); for the right-angled triangle BAP gives

$$AP = AB\cos BAP, \text{ or } r = 2c\cos\theta.$$

53. Let PY be a tangent to the circle at P (fig. 21), and AY a perpendicular upon it from the pole; and suppose $AY = p$, $AC = d$; then since $\angle\, YPC = 90^0$,

$$\frac{p}{r} = \sin APY = \cos APC = \frac{r^2 + c^2 - d^2}{2rc}, \text{ (Trig. Art. 93)},$$

or $2cp = r^2 + c^2 - d^2$, a relation between r and p.

If the pole be in the circumference, or $c = d$, this becomes

$$r^2 = 2cp.$$

54. From the preceding equations to the circle, which assume no other property of a circle than that it is the locus of a point which is always at the same distance from a given fixed point, all the theorems relative to the circle established in geometry, may readily be deduced. We shall however confine our attention to those which relate to the tangent.

Tangent and Normal to a Circle.

55. To find the equation to a straight line which shall touch a circle at a proposed point.

In geometry a line is said to touch a circle when it

has only one point in common with the circumference; if therefore through the two points P, P' (fig. 22), we draw a secant PP', and then make it turn about P, till P' coincides with P, the secant in its ultimate position will become a tangent at P, for it will have only one point in common with the circumference. This consideration furnishes an easy method of determining the tangent at a given point of the circumference.

Let x', y', be the co-ordinates of the given point P, and m the tangent of the angle which the touching line makes with the axis of x; then its equation will be

$$y - y' = m(x - x') \quad \text{(Art. 20)},$$

where m is to be determined.

Let x'', y'', be the co-ordinates of another point P' in the circle near the given point, and let α' be the angle which the line joining them makes with the axis of x; then (Art. 6),

$$\tan \alpha' = \frac{y' - y''}{x' - x''} = \frac{y'^2 - y''^2}{x'^2 - x''^2} \cdot \frac{x' + x''}{y' + y''} = -\frac{x' + x''}{y' + y''};$$

since, the points being in the circumference, their co-ordinates must satisfy the equation to the circle; and therefore

$$y'^2 = c^2 - x'^2, \; y''^2 = c^2 - x''^2, \text{ and } y'^2 - y''^2 = x''^2 - x'^2.$$

Now let $x'' = x'$, and $y'' = y'$, so that P' coincides with P, and the secant PT' assumes the position of the tangent PT; therefore, denoting by α the angle which the tangent forms with the axis of x, we get

$$m = \tan \alpha = -\frac{x'}{y'}, \text{ and } y - y' = -\frac{x'}{y'}(x - x')$$

for the equation to the tangent; or, multiplying by y', and observing that $x'^2 + y'^2 = c^2$, the equation, in its most simple form, becomes

$$yy' + xx' = c^2,$$

in which x', y', are the co-ordinates of the point of contact, and x, y, those of any point in the tangent line.

56. The equation to CP is $y = \frac{y'}{x'}x$, which compared with the above equation to PT, shews that CP and PT are at right angles to one another (Art. 23); that is, the tangent to a circle at any point, and the radius drawn to the point of contact, are perpendicular to one another.

Also the equation to the tangent in terms of its inclination to the axis of x, is

$$y = mx + \frac{c^2}{y'} = mx \pm c\sqrt{1+m^2}, \text{ since } c^2 = y'^2(1+m^2);$$

the lower sign referring to the point P''.

57. To find the equation to the normal at any point of a circle.

A line PG (fig. 22) drawn through the point of contact perpendicular to the tangent, is called a normal. The coordinates of the point being x', y', the equation to the normal will be of the form

$$y - y' = m'(x - x'), \text{ (Art. 20)};$$

and the condition of being perpendicular to the tangent gives

$$m' = -\frac{1}{m} = \frac{y'}{x'} \text{ (Art. 24)};$$

$$\therefore y - y' = \frac{y'}{x'}(x - x'), \text{ or, reducing, } y = \frac{y'}{x'}x,$$

which is the equation to a line passing through the origin, in this case the centre. Hence all normals to a circle pass through the centre.

58. To find the locus of the middle points of a system of parallel chords.

Let all the chords make an angle α' with axis of x, and let PP' be one of them (fig. 22) and x, y, the co-

ordinates of its middle point V; then proceeding as in Art. 55, and using the same notation,

$$2y = y' + y'', \quad 2x = x' + x'';$$

$$\therefore \frac{y}{x} = \frac{y' + y''}{x' + x''} = -\frac{1}{\tan a'},$$

and consequently the locus of V is a line through the centre perpendicular to the chords.

59. To find the equation to a straight line which shall touch a circle, and pass through a given point without the circle.

Let h, k, be the co-ordinates of the given point; and x', y', those of the point of contact, which are unknown; when they are found, we shall have, for the equation to the tangent,

$$yy' + xx' = c^2;$$

and as the tangent passes through the given point, its equation must be satisfied by the co-ordinates of that point,

$$\therefore ky' + hx' = c^2,$$

and since the point of contact is in the circumference,

$$y'^2 + x'^2 = c^2,$$

which are the two equations that serve to determine x' and y'.

It is evident that x' and y' will each have two values; therefore there will be two points of contact; and the equation

$$y'k + x'h = c^2,$$

since it is satisfied by the co-ordinates of the points, will be the equation (regarding x' and y' as the variable co-ordinates), to the chord joining the points of contact of two tangents drawn from the point (h, k); for if an equation of the first degree between two variables be satisfied by the co-ordinates of two points, it must be the equation to the straight line passing through those points.

60. When a problem, as in the present case, leads to two equations between the co-ordinates x' and y' of an unknown point, each of the equations, taken separately, gives a geometrical locus in which the point is placed; consequently, if we construct the two loci, we shall have two lines, the intersections of which will determine the points which satisfy the problem.

The locus of the second equation is the proposed circle; the locus of the first is a straight line AB (fig. 23), which is constructed by taking $CA = \frac{c^2}{h}$, $CB = \frac{c^2}{k}$, and joining AB; then the points T, T', in which this line cuts the circle are the points required.

Either of the equations may be replaced by another which results from combining them in any manner; and in solving problems in this way, we must always select the combinations whose loci are easiest to construct.

Thus, if we subtract the above equations, we get

$$y'^2 - y'k + x'^2 - x'h = 0, \text{ or } (y' - \tfrac{1}{2}k)^2 + (x' - \tfrac{1}{2}h)^2 = \tfrac{1}{4}k^2 + \tfrac{1}{4}h^2,$$

which represents a circle whose centre is O, the middle point of CP, and radius CO, P being the point through which the tangents are to be drawn; if then we join CP and bisect it in O (fig. 23), and with centre O and radius OC describe a circle cutting the former in T, T', these are the points of contact, and are determined by a simpler construction than the former one.

It is evident that the two tangents PT, PT', subtend equal angles at the centre.

The value $AC = \frac{c^2}{h}$, which determines the point A in which TT' meets the axis of x, is independent of the ordinate k of P; therefore A will remain in the same position, for all positions of P in the indefinite line PM parallel to the axis of y. If therefore from the several points of

any straight line (since the direction of the axis of y is arbitrary), we draw pairs of tangents to a circle and join the corresponding points of contact, all the secants will intersect in the same point; and conversely if through any point we draw different chords and apply two tangents at the extremities of each, the locus of the intersection of each pair of tangents will be a straight line.

SECTION V.

ON THE DIFFERENT ORDERS OF CURVES; AND ON THE DIVISION OF CONIC SECTIONS, OR CURVES OF THE SECOND ORDER, INTO THREE SPECIES.

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61. LINES are divided into orders according to the degree of their equations, the degree being determined by the sum of the indices of x and y in that term of the equation (which is supposed to contain no fractional or irrational term) where it is greatest.

The straight line is the line of the first order, being the locus of the equation of the first degree between two variables; the circle is a line of the second order, or curve of the second order (these terms being used indifferently), because its equation is of the second degree.

62. Curves of the second order are those whose equations involve the squares, or the simple product of the variables x and y; but no powers or products of them which are of higher dimensions. Hence the equation to curves of the second order under its most general form, or, which is the same thing, the general equation of the second order between two variables, is

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

which (as will be hereafter shewn) by giving a proper position and direction to the origin and axes of the co-ordinates, can always be reduced to one of the forms

$$Ay^2 + Bx^2 = C,$$

$$y^2 = Ax,$$

representing two distinct families of curves; the former those which have a centre, the latter those which have not a centre.

63. The centre of a curve is a point such that all lines drawn through it and meeting the curve both ways, are bisected in it.

An axis of a curve is a line with respect to which the curve is symmetrically situated.

64. Of the curves represented by the equation

$$Ay^2 + Bx^2 = C,$$

the origin is the centre, and the axes of the co-ordinates are axes.

For suppose P (fig. 24) to be a point in the curve having known co-ordinates x', y'; then the equation is satisfied by these values; and since it contains only even powers of x and y, it is also satisfied by the same values taken negatively; but if we produce PC to P' and make $CP' = CP$, the co-ordinates of P' are $-x'$ and $-y'$; therefore P' is a point in the curve, and PP' is a chord, and it is bisected in C; that is, every chord is bisected in C, and therefore C is the centre of the curve.

Also the curve is situated symmetrically with respect to the co-ordinate axes; for if in the equation we put $x = CN = x'$, we get for y two equal values with contrary signs, PN, P_1N; so that for every point situated above the axis of x, there will be a corresponding point situated at an equal distance below that axis. Similarly, for a given value $CM = y'$ of the ordinate, the equation furnishes two equal values with contrary signs, MP, MP'', of the abscissa. Hence each of the co-ordinate axes bisects its ordinates at right angles; and the curve is situated symmetrically with respect to them, or they are axes of the curve.

65. In the equation to curves of the second order that have a centre,

$$Ay^2 + Bx^2 = + C,$$

having taken care to make the second member positive, since the coefficients of the variables cannot be both nega-

tive together, we can have only two varieties of form; one with both coefficients positive, the other with one coefficient negative; so that the equation may assume the two forms

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The curves represented by these are called respectively, the Ellipse, and the Hyperbola.

In the particular case of $C = 0$, the equation may assume either the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$, which can only be satisfied by the values $x = 0$, $y = 0$, representing a point, viz. the origin; or the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = 0$ representing two straight lines; for the equation will evidently be satisfied by the co-ordinates of any point in either of the lines

$$\frac{x}{a} + \frac{y}{b} = 0, \quad \frac{x}{a} - \frac{y}{b} = 0.$$

66. Of the curve represented by the equation

$$y^2 = Ax,$$

the origin is not the centre, since the equation does not remain unaltered when x and y are changed into $-x$ and $-y$; and we shall see hereafter that it cannot have a centre; also, since only an even power of y enters into the equation, the axis of x is an axis of the curve, but the axis of y is not an axis of the curve.

The equation may always be reduced to the form $y^2 = 4ax$, where a is a positive quantity; because if A be negative, we have only to change x into $-x$, the effect of which will be merely to reverse the position of the curve. Hence the second division of lines of the second order offers only one variety, which is called the Parabola. If $A = 0$, the equation becomes $y^2 = 0$, representing a straight line, viz. the axis of x.

These three species of curves, to one or other of which all lines of the second order belong, are called Conic Sections.

67. Instead of entering upon the discussion of the general equation of the second order (which may more conveniently be reserved for a more advanced part of the work), we shall now separately investigate the equation to each of the Conic Sections from a simple definition which embraces all of them; and thence determine their figures and properties.

68. Def. The locus of a point whose distances from a given fixed point and a straight line given in position, are always to one another in a constant ratio, is called a Conic Section.

Thus let S (fig. 25) be the given fixed point, and KX the line given in position, P a point such that joining SP and drawing PM perpendicular to KX, the distances SP, PM, are always to one another in an invariable ratio, then the locus of P is a Conic Section.

The point S is called the focus, and the line KX the directrix. Since SP may be always equal to PM, or always less than PM in a constant ratio, or always greater than PM in a constant ratio, there will be a distinct species of Conic Section corresponding to each of these cases; in the first case the locus of P is called the Parabola, in the second the Ellipse, and in the third the Hyperbola.

The condition of SP being equal to, or less than PM, can only be satisfied when P falls on the same side of KX with S; but that of SP being greater than PM, may evidently be fulfilled, on whichever side of KX, P is taken. Therefore the Parabola and Ellipse will lie entirely on the same side of the directrix, as the focus does, by which they are described; but the Hyperbola will lie on both sides of the directrix.

SECTION VI.

ON THE PARABOLA.

Various Forms of the Equation to the Parabola.

69. To find the equation to the parabola.

The parabola is the locus of a point, whose distance from a given point is always equal to its distance from a given fixed line.

Let KX (fig. 26) be the given fixed line, and S the given point, from which draw SX perpendicular to KX, and bisect it in A; then A is a point in the curve; and since the distance SX is known, let it equal $2a$, and consequently $AS = a$.

Draw Ay parallel to KX, and take A for the origin, and Ax, Ay, for the rectangular axes of the co-ordinates; and let P be a point in the parabola, on the same side of KX as S, and $AN = x$, $PN = y$, its co-ordinates; then drawing PM perpendicular to KX, and joining SP,

$$SP^2 = PM^2,$$

$$\text{or } PN^2 + SN^2 = XN^2,$$

$$\text{or } y^2 + (x - a)^2 = (x + a)^2;$$

$$\therefore\ y^2 = 4ax,$$

the equation required.

70. To trace the parabola by means of its equation (fig. 26).

Solving the equation, we get $y = \pm 2\sqrt{ax}$,

which shews, since for each positive value of x there are two equal values of y with contrary signs, that the axis of x, Xx,

is an Axis of the curve, (Art. 63); and that the origin A is a point in the curve, since $x=0$, gives $y=0$; and that no part of the curve is situated to the left of A, for a negative value of x makes y imaginary; but that as x increases from zero to infinity, y also increases from zero to $\pm\infty$. Moreover, as we shall soon see, the tangent at the vertex is perpendicular to the axis; therefore about the vertex, and consequently at every point, the curve is concave toward its axis, otherwise it might be intersected in more than two points by a straight line, which is impossible, (as will appear, Art. 89). The parabola has only one vertex, namely, the point A where it is met by the axis; and only one focus and directrix; and consists of two perfectly similar infinite branches Ax, Ax', upon the same side of the axis of y, and situated symmetrically with respect to the axis of x, to which they turn their concavities.

71. The double ordinate through the focus is called the latus rectum of the parabola; to find its length, making $x = AS = a$, we get

$$y^2 = 4a^2; \quad \therefore\ y = \pm 2a = SB \text{ or } SC;$$

and consequently $BC = 4a$.

Hence, if P be any point in the curve, we have

$$PN^2 = BC \times AN;$$

or, the square of the ordinate is equal to the rectangle of the latus rectum and corresponding abscissa, and consequently varies as the abscissa.

Hence it is easy to determine any number of points in a parabola, whose latus rectum is known. Having taken $SX = \frac{1}{2}$ lat. rect. (fig. 26), through any point N in the axis erect the perpendicular PP'; then with centre S and radius equal to XN describe a circle cutting the perpendicular in P, P'; these are evidently points in the curve. To describe the parabola by a continuous motion, make a right-angled triangle KMR, having a string, length MR, fastened at R and the other end at S, slide along the directrix, and at the same time make a point P slide along RM so as always

to confine a portion of string PR against RM; then the point P will trace out a portion of the parabola, for SP will always equal PM. If the angle RMK were acute, the locus of P would be a hyperbola, because $SP = PM$ would be always greater, in a constant ratio, than the perpendicular distance of P from the line KX.

72. Let the origin be a point C (fig. 27) in the curve, and let the axis of the abscissæ be a line perpendicular to the axis of the parabola, passing through C; and let $CM = x$, $MP = y$, be the co-ordinates of any point P, and $CB = h$, $AB = k$, those of the vertex; then

$$PN^2 = 4a . AN,$$

$$\text{or } (h - x)^2 = 4a(k - y), \text{ or } -2hx + x^2 = -4ay,$$

$$\text{since } h^2 = 4ak; \text{ or } y = \frac{h}{2a}x - \frac{x^2}{4a},$$

another form of the equation which is sometimes useful.

73. To express the distance of any point in the parabola from the focus, in terms of its abscissa.

By Definition, $SP = PM$ (fig. 26),

$$= XN = XA + AN,$$

$$\therefore SP = x + a.$$

74. In expressing, as above, the distance of any point in the parabola from an assumed fixed point, it is only when the latter coincides with the focus that the expression becomes rational.

For let x', y', be the co-ordinates of the assumed point; x, y, those of the point in the parabola, and d their distance; then

$$d^2 = (x - x')^2 + (y - y')^2 = x^2 - 2xx' + x'^2 + y^2 - 2yy' + y'^2;$$

but $y = 2\sqrt{ax}$, therefore d^2, and *a fortiori* d, cannot be rational in terms of x, unless the term $2yy'$ vanish, which gives $y' = 0$; then, replacing y^2 by its value,

$$d^2 = x^2 + 4ax - 2xx' + x'^2,$$

which must be a perfect square;

$$\therefore\ 4x'^2 = 4(2a - x')^2,$$
$$x' = 2a - x',$$
$$x' = a,$$

which, with the value $y' = 0$, indicates the focus; this then is the only point whose distance from every point of the curve can be expressed rationally in terms of the abscissa or rather of the co-ordinates of the point. For, relative to any origin and axes whatever, we should have $SP = a + mx + ny + h$ (Art. 43).

This is sometimes given as the definition of the focus. If we take the focus for origin, and suppose the co-ordinate axes to be moved through an angle θ, we get

$$\sqrt{x'^2 + y'^2} = 2a + x' \cos\theta - y' \sin\theta;$$

which shews that the equation $x^2 + y^2 = (c + mx + ny)^2$ belongs to a parabola of which the origin is the focus, provided $m^2 + n^2 = 1$.

75. To find the polar equation to the parabola, the focus being the pole.

Let $SP = r$, $\angle PSx = \theta$, (fig. 26) be the polar co-ordinates of any point P; then

$$SP = PM = XS + SN,$$
$$\text{or } r = 2a + r\cos\theta,$$
$$\therefore\ r(1 - \cos\theta) = 2a;$$
$$\text{or } r = \frac{2a}{1 - \cos\theta}.$$

Sometimes the angle of revolution is measured from that part of the axis which passes through the vertex; in which case, if $ASP = \theta'$, putting $\pi - \theta'$ for θ, and therefore $-\cos\theta'$ for $\cos\theta$, we get

$$r = \frac{2a}{1 + \cos\theta'}.$$

Tangent and Normal to the Parabola.

76. To find the equation to the tangent of a parabola at a given point (fig. 28.).

We shall regard the tangent as a secant which passes at first through two points of the curve, and then turns about the given point till the other moves up to and coincides with it; there will be thus determined one definite position of the line which meets the curve without cutting it; for if, after P' has moved down to P, the line turned further about P in the same direction, it would cut the curve again below P. If m be the tangent of the angle which it ultimately makes with the axis of x, and x', y', the co-ordinates of the given point P (fig. 28) its equation will be

$$y - y' = m(x - x'), \quad \text{(Art. 20)},$$

where m is to be found in terms of x' and y'.

Let x'', y'', be the co-ordinates of another point P' in the parabola near the given point; if we draw a secant through these two points, and denote by α' the angle which it forms with the axis of x, we have (Art. 6),

$$\tan \alpha' = \frac{y'' - y'}{x'' - x'}.$$

But the points being in the parabola, we have

$$y''^2 = 4ax'', \quad y'^2 = 4ax';$$

$$\therefore \; y''^2 - y'^2 = 4a(x'' - x'),$$

$$\frac{y'' - y'}{x'' - x'} = \frac{4a}{y'' + y'},$$

$$\tan \alpha' = \frac{4a}{y'' + y'}.$$

Now let P' move up to and coincide with P, then $x'' = x'$, $y'' = y'$, and the secant becomes the tangent at (x', y'); there-

fore, denoting by α the angle PTN which the tangent makes with the axis of x, we get

$$m = \tan\alpha = \frac{4a}{2y'} = \frac{2a}{y'};$$

and consequently the equation to the tangent is

$$y - y' = \frac{2a}{y'}(x - x'),$$

or, multiplying by y' and observing that $y'^2 = 4ax'$,

$$yy' = 2a(x + x');$$

in which x', y', are the co-ordinates of the point of contact, and x, y, co-ordinates of any point in the tangent line; or, lastly, in terms of its inclination to the axis, the equation to the tangent is

$$y = mx + \frac{2ax'}{y'} = mx + \tfrac{1}{2}y' = mx + \frac{a}{m}.$$

77. If in the formula $\tan\alpha = \dfrac{2a}{y'}$, we make $y' = 0$, we find $\tan\alpha = \infty$; therefore the tangent at the vertex is perpendicular to the axis. Also if we suppose y' to increase up to infinity, α decreases to zero; therefore the tangent to the parabola continually tends to become parallel to the axis.

Hence, the equation of Art. 72 may be put under the form

$$y = x\tan\beta - \frac{x^2}{4a},$$

putting $\angle\, TCB = \beta$ (fig. 27); for $\cot TCB = \dfrac{2a}{h}$.

78. In any curve the distance between the foot of the ordinate to any point, and the intersection of the tangent at that point with the axis, is called the subtangent.

In the parabola, the subtangent is double of the abscissa.

For $TN \times \tan PTN = PN$ (fig. 29),

$$\text{or (Art. 76) } TN \times \frac{2a}{y'} = y'; \quad \therefore \ TN = \frac{y'^2}{2a} = 2x' = 2AN.$$

This result may also be obtained by making $y = 0$ in the equation to the tangent; this gives $x = -x'$, and proves that the point T where the tangent meets the axis of x, is situated to the left of A, and at a distance $AT = AN$. Hence, adding AN to AT, we have the subtangent $TN = 2AN$.

This property furnishes a simple construction for drawing a tangent to a parabola at a given point of the curve.

If P be the given point, and AN, NP, its co-ordinates, we have only to take in NA produced, $AT = AN$, and join TP, then TP is the tangent required.

79. In any curve, a line drawn through the point of contact perpendicular to the tangent is called a normal; and the distance between the foot of the ordinate, and the intersection of the normal with the axis of x, is called the subnormal.

In the parabola, the subnormal is equal to half the latus rectum.

For if PG be perpendicular to PT, since in the triangle TPG (fig. 29), PN is drawn from the right angle perpendicular to the opposite side,

$$NG \times TN = PN^2, \quad \text{or } NG \times 2x = 4ax;$$

$$\therefore \ NG = 2a = \text{half the latus rectum.}$$

80. This result may also be obtained by finding the equation to the normal at the point (x', y') of the parabola.

It will be of the form $y - y' = m'(x - x')$; and since the normal is perpendicular to the tangent whose equation is

$$y - y' = \frac{2a}{y'}(x - x'), \text{ (Art. 76),}$$

we have $m' = -\frac{1}{m} = -\frac{y'}{2a}$ (Art. 23);

therefore the equation to the normal is

$$y - y' = -\frac{y'}{2a}(x - x');$$

now make the ordinate of the normal $y = 0$, then $x - x' = 2a$,

or $AG - AN = NG = 2a$.

Also since $y' = -2am'$, $x' = \frac{y'^2}{4a} = am'^2$, the equation to the normal in terms of its inclination to the axis, is
$y + 2am' = m'(x - am'^2)$.

81. Since $ST = AS + AT = a + x$,

and $SG = SN + NG = x - a + 2a = a + x$,

we have $SP = ST = SG$ (Art. 73).

Draw Px' through P parallel to the axis,

then $\angle tPx' = PTS = SPT$, since $SP = ST$;

also $\angle GPx' = SPG$.

Hence the tangent and normal at any point make equal angles with the focal distance of that point, and with a line drawn through it parallel to the axis.

82. These properties furnish a simple method of drawing a tangent to a parabola through a given point.

First, let the point be in the parabola, as P (fig. 29); join SP, and with centre S and radius SP, describe a circle cutting the axis in T and G; then if PT and PG be joined, they are the tangent and normal at P.

83. Next, let the point be without the parabola, as T (fig. 30); and with centre T and radius TS describe a circle cutting (as it necessarily must, since T is nearer to the

directrix than to the focus) the directrix in two points M and M', through which draw two parallels to the axis, meeting the parabola in P, P', these are the points of contact; for the triangles MPT, TPS, are equal in all respects, therefore $\angle TPM = TPS$, and PT is a tangent at P; similarly, TP' is a tangent at P'.

It may be observed that the tangents TP, TP', subtend equal angles at S; for $\angle TMP = \angle TM'P'$, being complements of the equal angles TMM', $TM'M$; therefore, $\angle TSP = TSP'$. Also $\angle SPT = TPM = \text{compl}^{t}.$ of $PMS = SMM' = STP'$; therefore the triangles STP, STP' are similar; hence

$$ST^2 = SP \,.\, SP', \text{ and } \frac{TP^2}{TP'^2} = \frac{\Delta STP}{\Delta STP'} = \frac{SP}{SP'}.$$

84. The problem of drawing tangents to a parabola from an external point, may be also solved by means of the equation to the tangent, as in the case of the circle.

Let h, k, be the co-ordinates of the given external point, and x', y', those of the unknown point of contact; then since x' and y' must satisfy both the equation to the tangent and that to the curve, we have, to determine them,

$$ky' = 2a(h + x'), \quad y'^2 = 4ax';$$

and if we construct the straight line represented by the former, considering x' and y' as the variables, the points in which it intersects the parabola are the points of contact.

Hence it follows that if a pair of tangents to a parabola be drawn from an external point (h, k), the equation to the chord joining the points of contact is

$$ky = 2a(x + h);$$

for it is the equation to a line which determines, by its intersection with the parabola, the points of contact.

To find the angle α between the tangents that intersect

in a given point we have, if m, m', be the tangents of the angles which the touching lines make with the axis,

$$(1 + mm')^2 \tan^2 \alpha = (m - m')^2 = (m + m')^2 - 4mm',$$

$$\text{or } \left(1 + \frac{a}{h}\right)^2 \tan^2 \alpha = \left(\frac{k}{h}\right)^2 - \frac{4a}{h},$$

since m, m', are the roots of $m^2 h - mk + a = 0$, (Art. 76).

If α be invariable, then $(a + h)^2 \tan^2 \alpha = k^2 - 4ah$, or $(a + h)^2 \sec^2 \alpha = k^2 + (h - a)^2$, is the equation to the locus of the intersection of two tangents to a parabola that include a constant angle, and represents a hyperbola with the same focus.

85. The locus of the foot of the perpendicular dropped from the focus upon the tangent to a parabola, is the line touching the parabola at its vertex.

Let PT (fig. 29) the tangent at P, meet Ay, the line touching the parabola at its vertex in Y, and join SY; then because TN is bisected in A, PT is bisected in Y since Ay and PN are parallel (Art. 77); and since $SP = ST$, the triangles SPY, STY, are equal in all respects; therefore SY is perpendicular to PT. Hence the tangent at any point, and the perpendicular upon it from the focus, intersect in the line which touches the parabola at the vertex.

Also, from the right-angled triangle SYT, since AY is drawn from the right-angle perpendicular to the opposite side, we have

$$SY^2 = ST \times SA = SP \times SA,$$

or $p^2 = ar$, denoting SP, SY, by r and p respectively.

86. Let the tangent at P (fig. 31) meet the directrix in Q, draw PM perpendicular to the directrix, and join SQ; then $SP = PM$, PQ is common to the two triangles SPQ, QPM, and $\angle SPQ = QPM$ by what has been proved;

$$\therefore \angle QSP = PMQ = \text{a right angle.}$$

Hence if a perpendicular through the focus to any focal distance SP, intersect the directrix in Q, and QP be joined, QP is a tangent at P.

Therefore, producing PS to meet the parabola in P' and joining QP', this is a tangent at P'; and since $\angle PQS = PQM$ and $\angle P'QS = P'QM'$, we have $\angle PQP' = \frac{1}{2}\pi$. Hence the tangents at the extremities of any focal chord intersect at right angles in the directrix; and the line joining their point of intersection and the focus, is perpendicular to the chord.

87. Any circle passing through the points of intersection of three tangents to a parabola, will also pass through the focus.

Let P, Q, P', (fig. 77) be the three points of contact, L, M, N, the three points of intersection. Draw SD, SE, SF, from the focus to the points where the tangents cut the tangent at the vertex; these are respectively perpendicular to the tangents. Then

$$\angle LMN = DTS + FRS = SDE + SFE$$
$$= SLE + SNE = \pi - LSN;$$

therefore a circle may be described about the quadrilateral $MLSN$.

88. The results in Arts. 85 and 86 may also be obtained as follows. The equation to the tangent at (x', y') being

$$y = mx + \frac{a}{m}, \text{ where } m = \frac{2a}{y'},$$

the equation to a perpendicular upon it from the focus is

$$y = -\frac{1}{m}(x - a);$$

hence the co-ordinates of the point of intersection are $x = 0$, which characterises the tangent at the vertex, and

$$y = \frac{a}{m} = \frac{y'}{2};$$

$$\therefore\ SY^2 = a^2 + y^2 = a^2 + ax' = SP \,.\, SA.$$

Next let h, k be the co-ordinates of the point of intersection of a pair of tangents, then the chord joining the points of contact has for its equation

$$ky = 2a(x + h);$$

and if it pass through the focus, when $x = a$, $y = 0$;

$$\therefore\ h = -a, \text{ the equation to the directrix.}$$

Also the equation to a perpendicular to the chord through the focus, is

$$y = -\frac{k}{2a}(x - a); \text{ and when } x = -a,\ y = k;$$

therefore the perpendicular passes through the intersection of the tangents.

Lastly, let x', y', x'', y'', be the co-ordinates of the extremities of the focal chord, then they satisfy its equation;

$$\therefore\ \frac{y'}{y''} = \frac{x' - a}{x'' - a} = \frac{y'^2 - 4a^2}{y''^2 - 4a^2}, \text{ or } y'y'' + 4a^2 = 0;$$

consequently the tangents at the extremities of the focal chord are perpendicular to one another.

The Parabola referred to Oblique Co-ordinates.

89. To determine the intersection of a straight line with a parabola.

Let $y = mx + c$, be the equation to any straight line; if this line intersect a parabola whose equation is $y^2 = 4ax$, and if x', y', be the co-ordinates of a common point, we have

$$y' = mx' + c, \quad y'^2 = 4ax';$$

therefore, substituting $\frac{1}{m}(y' - c)$ for x' in the latter, we get

$$y'^2 - \frac{4a}{m}y' + \frac{4ac}{m} = 0,$$

the roots of which are the ordinates of the points where the straight line meets the curve, and the abscissæ are known from $x' = \frac{1}{m}(y' - c)$. Hence if the roots are real, the straight line will cut the parabola in two points, and it cannot cut the parabola in more than two points; if the roots are imaginary, the line falls entirely without the parabola.

If the roots be equal, the points of section coincide, and the line is then a tangent; and we have, since the first member of the equation is a perfect square,

$$\frac{16ac}{m} = \frac{16a^2}{m^2}, \quad \text{or } c = \frac{a}{m};$$

$$\therefore \; y = mx + \frac{a}{m},$$

is the equation to the tangent to a parabola, in terms of the angle which it makes with the axis, agreeably to Art. 76.

90. To find the locus of the middle points of a system of parallel chords.

Let QQ' (fig. 32) be any chord whose equation is $y=mx+c$, V its middle point; draw VM perpendicular to Ax, then

$$VM = Q'N' + \tfrac{1}{2}(QN - Q'N') = \tfrac{1}{2}(Q'N' + QN);$$

but the values of QN, $Q'N'$, are the roots of the equation

$$y^2 - \frac{4a}{m}y + \frac{4ac}{m} = 0,$$

obtained by eliminating x between the equations to the parabola and chord, as in Art. 89;

$$\therefore \; QN + Q'N' = \frac{4a}{m};$$

hence, denoting the ordinate of V by Y, we have for the equation to the locus of V, $Y = \frac{2a}{m}$, which represents a straight line PV parallel to the axis; and since the equation

does not involve c, PV bisects all chords for which m is the same, that is, the system of chords parallel to QQ'.

91. The line PV is called a diameter of the parabola; and the semi-chord QV is called an ordinate to the diameter PV.

Hence all diameters of a parabola are straight lines parallel to the axis; and conversely, every straight line parallel to the axis may be considered as a diameter of the parabola; for by giving m a suitable value in the equation $Y = \frac{2a}{m}$, Y may become equal to any quantity we please.

92. Suppose a diameter Px' to be drawn at a distance y' from the axis; then we have for this diameter

$$\frac{2a}{m} = y', \quad \text{or } m = \frac{2a}{y'}.$$

The quantity m is the tangent of the angle at which the diameter in question meets the chords which it bisects; it is also (Art. 76) the value of the tangent of the angle which the line touching the parabola at P, makes with the axis of x; therefore the chords bisected by any diameter, are parallel to the tangent at the extremity of that diameter; as might have been foreseen; for of the parallel chords which PV bisects, that which is indefinitely near to P, will ultimately coincide in direction with the tangent at P. Hence, also, the diameters bisect the corresponding chords at different angles varying from a right angle to zero. (Art. 77).

In order that m may be infinite, we must have $y' = 0$; hence the axis of x is the only diameter which bisects its ordinates at right angles, or is the only axis of the parabola.

93. To find the equation to the parabola when referred to the system of oblique axes formed by any diameter, and the tangent at the extremity of the diameter.

Suppose the new origin to be a point P (fig. 32) in the curve, and h and k its co-ordinates; then between h and k

we have the relation $k^2 = 4ah$; also let the diameter Px' be the axis of x', and the tangent at its extremity, Py', the axis of y'; and let $\angle y'Px' = \alpha$, then $\tan\alpha = \dfrac{2a}{k}$ (Art. 76). Let $AN = x$, $NQ = y$, be the co-ordinates of any point Q reckoned from the vertex as origin; and $PV = x'$, $QV = y'$, its co-ordinates referred to the new axes; then $VR = y'\cos\alpha$, $RQ = y'\sin\alpha$, and

$$x = AM + MN = h + x' + y'\cos\alpha,$$

$$y = NR + RQ \ = k + y'\sin\alpha.$$

Now substituting in the equation $y^2 = 4ax$, we get

$$(y'\sin\alpha + k)^2 = 4a(x' + y'\cos\alpha + h),$$

$$\text{or } y'^2\sin^2\alpha + 2y'k\sin\alpha + k^2 = 4ax' + 4ay'\cos\alpha + 4ah,$$

or $y'^2\sin^2\alpha = 4ax'$, since $k^2 = 4ah$, and $k\sin\alpha = 2a\cos\alpha$.

$$\text{But (Trig. 18) } \frac{a}{\sin^2\alpha} = a\,\text{cosec}^2\alpha = a(1 + \cot^2\alpha)$$

$$= a\left(1 + \frac{k^2}{4a^2}\right) = a + h = SP,$$

$$\therefore\ y'^2 = 4SP\,.\,x' = 4a'x', \text{ if } SP = a',$$

$$\text{or } QV^2 = 4SP\,.\,PV.$$

94. The coefficient $4SP$, by which one diameter differs from another, is called the *parameter* of the diameter to which the parabola is referred; it is equal to four times the distance of the focus from the extremity of the diameter. It is also equal to the double ordinate passing through the focus. For draw QQ' (fig. 33) through the focus S parallel to the tangent PT; then $PV = ST = SP$,

$$\therefore\ QV^2 = 4SP \times PV = 4SP^2;$$

$$\therefore\ QV = 2SP, \text{ and } QQ' = 4SP.$$

95. The equation to a parabola being of the same form when referred to a diameter and the tangent at its

extremity, as when referred to the axis of the parabola, the properties which are independent of the inclination of the co-ordinates, will be the same in the two systems. Hence, taking $y^2 = 4a'x$ for the equation to the parabola referred to the oblique axes Px', Py', (fig. 35), the equation to the tangent at a point $Q(x', y')$ will be (Art. 76)

$$yy' = 2a'(x + x'),$$

where $\frac{2a'}{y'}$ denotes the ratio of the sines of the angles which the tangent makes with the axes of x and y (Art. 30); and when the tangent meets the axis of x, we shall have $x = -x'$, or $PT = PV$; i.e. the subtangent equals twice the abscissa of the point of contact, in all cases.

96. Also, if we wish to draw a tangent through an external point $Q(h, k)$ (fig. 34), we have, to determine the points of contact (x', y'), the equations

$$y'^2 = 4a'x', \quad y'k = 2a'(x' + h);$$

the latter, considering x' and y' as the variables, being the equation to the chord joining the points of contact; and if this line be constructed by taking

$$PT = -h, \quad PR = \frac{2a'h}{k},$$

and joining TR, it will cut the parabola in the two points of contact.

97. Since the distance PT is independent of k, if through Q we draw a line parallel to Py', and from any other point in this line we draw a pair of tangents to the parabola, the secant passing through the new points of contact will cut the diameter Px' in T, as this point only changes when h changes. Hence if from the several points of any straight line, pairs of tangents be drawn to a parabola, the secants joining the corresponding points of contact will all intersect in the same point; and conversely, if through any point we draw different chords, and draw two tangents at

the extremities of each, the locus of the intersection of the tangents will be a straight line.

Hence it appears that the same equation $ky = 2a(x + h)$ represents (1) the tangent at the point (h, k) of the curve; (2) the chord of contact of two tangents drawn from an external point (h, k); and (3) the locus of the intersection of pairs of tangents applied at the extremities of all chords passing through any point (h, k).

98. The tangents at the extremities of any chord will intersect in the diameter of which the chord is an ordinate.

For taking the diameter and the tangent at its extremity as axes, the equation to the tangent will be

$$\pm yy' = 2a'(x + x');$$

using the upper or lower sign, according as we consider the point $Q(x', y')$, or the other extremity of the chord Q', whose co-ordinates are x', $-y'$, to be the point of contact (fig. 35); and in both cases $y = 0$ when $x = -x'$; therefore the tangents meet the diameter in the same point T.

99. Having given the parameter of any diameter of a parabola, and the inclination of the corresponding ordinates, to describe the parabola.

Let Px' be the given diameter (fig. 33); draw the line $y'PT$ at the given inclination to Px', this line will be a tangent to the parabola at the point P. Make the angle $TPS = y'Px'$, and $PS =$ a quarter of the given parameter; then S will be the focus. In PV produced backwards take $PM = PS$, and draw ML perpendicular to Mx', this will be the directrix; and the focus and directrix being known, the parabola can of course be described.

100. If a parabola be traced upon a plane, we may determine its axis by drawing two parallel chords QQ', qq' (fig. 35), and drawing a line VV' through their middle points, this will be a diameter. And if we draw any chord QR perpendicular to it, and through the middle point of QR

draw AN parallel to VV', this will be the axis of the parabola, and if from P we draw a line making with the tangent at P an angle equal to $y'Px'$, it will intersect the axis in S the focus.

101. If through any point within or without a parabola, two lines be drawn parallel to two given straight lines to meet the curve, the rectangles of the segments will be to one another in an invariable ratio.

Let O be the given point (fig. 36), Qq a line drawn through it in a known direction, and therefore an ordinate to a given diameter PV; draw the diameter $A'O$, and $A'v$ parallel to Qq; then

$$QO \times Oq = QV^2 - VO^2 = 4SP \times PV - 4SP \times Pv = 4SP \times A'O.$$

Similarly, if $Q'q'$ be an ordinate to the diameter whose extremity is P', passing through O,

$$Q'O \times Oq' = 4SP' \times A'O;$$

$$\therefore\ QO \times Oq : Q'O \times Oq' :: SP : SP',$$

a ratio independent of the position of O.

If one of the lines be parallel to the axis, then it is the ratio $QO \times Oq : A'O$ that is invariable, however Qq and $A'O$ move parallel to themselves.

102. Hence if we suppose Qq, $Q'q'$, to move parallel to themselves till they become tangents to the parabola at the points P and P', and intersect in a point O without the curve, we have agreeably to Art. 83,

$$OP^2 : OP'^2 :: SP : SP'.$$

103. Any parabolic segment ANP cut off by a diameter and its semi-ordinate, is two-thirds of the parallelogram whose sides are the abscissa AN and the ordinate NP.

Let NP, $N'P'$ (fig. 37), be ordinates to the diameter AN; at P, P', draw tangents meeting the diameter in T,

T', and one another in R. Join PP' and draw RK parallel to AN, meeting PP' in I, and bisecting it in that point (Art. 98); and draw KH perpendicular to AN.

$$\text{Then area of triangle } TRT' = \tfrac{1}{2} TT' \times KH,$$

$$\text{area of trapezium } N'P'PN = NN' \times KH = TT' \times KH,$$

$$\text{(Art. 35) since } AN = AT, \text{ and } AN' = AT'.$$

Hence the trapezium is double of the triangle. Similarly we may shew that trapezium $N''P''P'N'$ is double of the triangle $T'R'T''$, and so on. Hence the sum of the trapeziums is double of the sum of the triangles; and therefore the parabolic segment APN, which is the limit of the first sum, is double of the exterior segment APT, which is the limit of the second sum. Hence the segment ANP is two-thirds of the triangle TNP, or two-thirds of the parallelogram contained by AN, NP.

SECTION VII.

ON THE ELLIPSE.

Various Forms of the Equation to the Ellipse.

104. To find the equation to the Ellipse.

The ellipse is the locus of a point, whose distance from a given point is always less than its distance from a given fixed line, in a constant ratio.

Let S (fig. 38) be the given point, and KK' the given fixed line; and from S let fall the perpendicular SX upon KK'. Let P be a point in the ellipse; join SP and draw PM perpendicular to KX, and let the constant ratio of SP to PM be $e : 1$, e being less than 1; then P is on the same side of KK' as S. Divide SX in A so that $SA : AX :: e : 1$, then A is a point in the ellipse; and since the distance SX is known, we may assume $AS = p$, therefore $AX = \frac{p}{e}$. Through A draw Ay parallel to KX, and take A for the origin, and Ax, Ay, for the rectangular axes of co-ordinates; and let $AN = x$, $PN = y$, be the co-ordinates of P, so that $SN = x - p$, $XN = x + \frac{p}{e}$.

$$\text{Then } SP^2 = e^2 . PM^2,$$

$$\text{or } SN^2 + NP^2 = e^2 . XN^2,$$

$$\text{or } (x - p)^2 + y^2 = e^2\left(\frac{p}{e} + x\right)^2 = (p + ex)^2;$$

$$\therefore y^2 = 2p(1+e)x - (1 - e^2)x^2$$

$$= (1 - e^2) . \left(\frac{2p}{1-e}x - x^2\right);$$

or, if we replace the known quantity $\frac{p}{1-e}$ by a,

$$y^2 = (1-e^2)(2ax - x^2),$$

the required equation.

105. To determine the points where the curve cuts the axis of x, make $y = 0$ in this equation; then $x = 0$ or $x = 2a$; the value $x = 0$, gives the point A already known; the other value $x = 2a = AA'$, determines the point A'.

Bisect AA' in C, then making in the equation to the ellipse $x = AC = a$, we get

$$y^2 = (1-e^2)\,a^2, \quad \text{or } y = \pm a\sqrt{1-e^2}.$$

If therefore through C we draw BB' perpendicular to AA', and take $CB = CB' = a\sqrt{1-e^2}$; B, B', are points in the ellipse; and denoting BB' by $2b$, we have

$$a\sqrt{1-e^2} = b \text{ and } 1 - e^2 = \frac{b^2}{a^2},$$

and the equation to the ellipse becomes

$$y = \pm \frac{b}{a}\sqrt{2ax - x^2}.$$

106. In order to transfer the origin to C, since $AN = AC + CN$, we must change x into $a + x'$; therefore the equation to the ellipse referred to its center and axes, becomes

$$y^2 = \frac{b^2}{a^2}\left\{2a(a + x') - (a + x')^2\right\} = \frac{b^2}{a^2}(a^2 - x'^2),$$

or, in slightly different forms, suppressing the accent,

$$a^2y^2 + b^2x^2 = a^2b^2, \text{ or } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This form of the equation shews that the origin is the center of the ellipse, and that the axes of the co-ordinates are Axes of the ellipse (Art. 64); but the term, axis, is more

particularly appropriated to the portions of those lines, $AA' = 2a$, $BB' = 2b$, which fall within the curve; of these the greater (which passes through the foci, as we shall see) is called the *major* axis, and sometimes the transverse axis; and the other the *minor*, or conjugate axis. Their extremities A, A', B, B', are called the vertices of the ellipse, and their intersection, as has been said, the center.

107. To trace the ellipse by means of its equation.

The equation to the ellipse referred to its center and axes is

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}.$$

Hence as x increases positively from zero to a, the two values of y are real, and diminish from b to zero, and give the portion of the curve $BA'B'$ (fig. 38); but when x exceeds a, the values of y become imaginary, and therefore no part of the curve lies beyond A'; also the curve in every one of its points must have its concavity turned towards the center, otherwise it might be cut by a straight line in more than two points, which is impossible, (as will appear Art. 137). Hence, since the portion of the curve situated to the left of BB', is precisely similar and equal to the portion situated to the right, the shape of the curve is that of the oval $ABA'B'$, surrounding the center on every side, and every point in it being at a finite distance from the center.

108. Since the ellipse is symmetrically situated with respect to the axes AA', BB', if we take $CH = CS$, $Cx = CX$, and draw kx perpendicular to AA', there is no reason why the curve may not be described by means of the focus H and directrix kx, just as well as by means of S and KX. Hence the ellipse has two foci S and H, equidistant from C.

Also, since (Art. 104) $a = \dfrac{p}{1-e}$, we have $AS = p = a(1-e)$;

$$\therefore SC = AC - AS = a - a(1-e) = ae, \text{ and } e = \frac{SC}{AC}.$$

109. The quantity e, which expresses the ratio of the distance between either focus and the center, to the semi-axis major, is called the eccentricity.

Since (Art. 105) $b = a\sqrt{1-e^2}$, the eccentricity e, expressed by the semi-axes, is equal to $\frac{\sqrt{a^2-b^2}}{a}$.

$$\text{Hence } SC = ae = \sqrt{a^2-b^2}, \text{ and } \therefore BS = a;$$

and if with center B and radius equal to the semi-axis major we describe a circle, it will intersect the major axis in the foci.

Hence also, each focus divides the major axis into the segments $a - \sqrt{a^2-b^2}$, $a + \sqrt{a^2-b^2}$, the product of which equals the square of the semi-axis minor; that is,

$$AS . A'S = BC^2.$$

110. Since $AS = e . AX$, we have $AX = \frac{a(1-e)}{e}$;

$$\therefore CX = CA + AX = \frac{a}{e} = \frac{a^2}{ae} = \frac{CA^2}{CS},$$

$$\text{and } SX = SA + AX = \frac{a(1-e^2)}{e} = \frac{CB^2}{CS};$$

which determine the directrix relative to the center, and focus.

111. The double ordinate passing through the focus is called the latus rectum.

To find its value, make $x = CH = ae$ in the equation

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2), \text{ (Art. 106)}$$

$$\text{then } y^2 = \frac{b^2}{a^2}(a^2 - a^2e^2) = b^2(1-e^2) = \frac{b^4}{a^2}; \text{ (Art. 105)}$$

$$\therefore y = \pm\frac{b^2}{a};$$

$$\therefore LL' = \frac{2b^2}{a}, \text{ or } = 2a(1-e^2).$$

112. If the distance SC between the focus and center of an ellipse be supposed to become infinite, the distance AS between the focus and vertex remaining finite, the ellipse will be changed into a parabola.

For since $\frac{b^2}{a^2} = 1 - e^2$, the equation to the ellipse reckoned from the vertex, may be written

$$y^2 = 2a(1 - e^2)x - (1 - e^2)x^2,$$

$$\text{or } y^2 = 2p(1 + e)x - (1 - e^2)x^2, \text{ if } AS = a(1 - e) = p.$$

But (Art. 108) $e = \frac{SC}{AC} = \frac{SC}{p + SC} = \frac{1}{1 + \frac{p}{SC}}$;

$$\text{let } SC = \infty; \quad \therefore e = 1;$$

$$\therefore y^2 = 4px, \text{ the equation to a parabola.}$$

Hence, if any result be obtained for the ellipse, we may, by this modification, adapt it to the parabola; that is, by expressing it in terms of AS and SC, and then making $SC = \infty$, the origin of co-ordinates being supposed to be at the vertex or focus.

113. When $a = b$, the equations to the ellipse become

$$y^2 = 2ax - x^2, \quad y^2 = a^2 - x^2, \text{ (Arts. 105, 106),}$$

which represent a circle; hence when its axes are equal, the ellipse becomes a circle.

Upon the major axis as diameter describe a circle (fig. 39), and produce the ordinate NP of the ellipse to meet it in Q; then making $CN = x$, $NP = y$, $NQ = y'$, we have

$$y = \frac{b}{a}\sqrt{a^2 - x^2}, \quad y' = \sqrt{a^2 - x^2};$$

$$\therefore y = \frac{b}{a}y';$$

which shews that, corresponding to the same abscissa, the ordinate of the ellipse is to the ordinate of the circle in the constant ratio of the smaller to the larger axis; consequently

the ellipse may be described by diminishing all the ordinates of the circle in that ratio.

114. Since $y^2 = \frac{b^2}{a^2}(a^2 - x^2) = \frac{b^2}{a^2}(a + x) \cdot (a - x)$,

$$\text{gives } PN^2 = \frac{b^2}{a^2} \times A'N \times AN \text{ (fig. 39)},$$

we see that the square of any ordinate is to the rectangle of the corresponding segments of the major axis, as the square of the semi-axis minor to the square of the semi-axis major; and that, consequently, the square of the ordinate varies as the rectangle of the corresponding segments of the major axis.

115. To express the distances of any point in the ellipse from the foci, in terms of its abscissa.

By Definition, (fig. 38),

$$SP = e \cdot PM = e\,(CX + CN) = e\left(\frac{a}{e} + x\right) \quad \text{(Art. 110)}$$

$$= a + ex.$$

$$HP = e \cdot PM' = e\,(Cx - CN) = e\left(\frac{a}{e} - x\right)$$

$$= a - ex;$$

since e is less than 1, and x is always less than a, this expression for HP is always positive.

If x be measured from S, then

$$SP = e \cdot PM = e\,(SX + SN) = a\,(1 - e^2) + ex.$$

116. In expressing, as above, the distance of any point in the ellipse from an assumed fixed point, it is only when the latter coincides with one of the foci, that the expression becomes rational in terms of the abscissa of the point.

For let x', y', be the co-ordinates of the assumed point, x, y, those of any point in the ellipse, and d their distance,

then $d^2 = (x - x')^2 + (y - y')^2$

$$= x^2 - 2xx' + x'^2 + y^2 - 2yy' + y'^2.$$

But $y = \frac{b}{a}\sqrt{a^2 - x^2}$, therefore d^2, and *a fortiori* d, cannot be expressed rationally in terms of x, unless the term $2yy'$ disappear, which gives $y' = 0$; then replacing y^2 by its value, we get

$$d^2 = \left(1 - \frac{b^2}{a^2}\right) x^2 - 2xx' + x'^2 + b^2,$$

which must be a perfect square;

$$\therefore\ 4\left(1 - \frac{b^2}{a^2}\right)(x'^2 + b^2) = 4x'^2,$$

$$\text{or } x' = \pm\sqrt{a^2 - b^2}.$$

These values require that a should be greater than b, i.e. that the abscissæ should be measured along the axis major; and with $y' = 0$, they determine, as we perceive, the two foci S and H. These then are the only points whose distances from every point of the curve can be expressed rationally in terms of the abscissa, or rather of the co-ordinates of the point. For, relative to any origin and axes whatever, we should have $SP = a + e(mx + ny + h)$.

This is sometimes given as the definition of the focus. If the co-ordinate axes were turned about the focus through an angle θ, the formula $\sqrt{x^2 + y^2} = a(1 - e^2) + ex$ would be transformed into

$$\sqrt{x'^2 + y'^2} = a(1 - e^2) + e(x'\cos\theta - y'\sin\theta) = c + mx' + ny'.$$

Hence we see that $x^2 + y^2 = (c + mx + ny)^2$ represents an ellipse with its major axis inclined to the axis of x at an angle whose tangent $= -\frac{n}{m}$, and of which the origin is one of the foci, provided $m^2 + n^2 < 1$.

117. Hence, by addition we get

$$SP + HP = 2a,$$

or, the sum of the focal distances of any point in the ellipse is constant, and equal to the major axis. Also, for a point not in the curve, the sum of the focal distances is greater or less than $2a$, according as the point is situated outside or inside the ellipse (Eucl. I. 21).

This property affords a simple method of determining any number of points of an ellipse of which we know the foci and axis major. In AA' take any point F (fig. 40), and with centre S and radius $A'F$ describe a circle; next with centre H and radius AF describe another circle, cutting the former in P, P', these are manifestly points in the ellipse.

When the ellipse is to be very large, we may describe it by fastening in the foci the ends of a cord of the same length as the axis major; then if we make a pole slide along the cord so as to keep it stretched, the ellipse will be traced out by the extremity of the pole.

This property also furnishes the following method of investigating the equation to the ellipse.

118. To find the locus of a point, the sum of whose distances from two given points is constant.

Through the two fixed points S, H (fig. 40), draw the indefinite line Sx; bisect SH in C, and draw yC perpendicular to it, and take Cx, Cy, for the axes of the co-ordinates, as the locus of P will evidently be symmetrical with respect to these lines. Let $SC = CH = c$, $CN = x$, $NP = y$, the co-ordinates of any point P, and $SP + HP = 2a$;

$$\text{then } SP^2 = (x + c)^2 + y^2,$$

$$HP^2 = (x - c)^2 + y^2;$$

$$\therefore\ SP^2 - HP^2, \text{ or } 2a(SP - HP) = 4cx;$$

$$\therefore\ SP - HP = \frac{2cx}{a},$$

$$\text{but} \quad SP + HP = 2a\,;$$

$$\therefore\; SP = a + \frac{cx}{a};$$

$$\therefore\; \left(a + \frac{cx}{a}\right)^2 = (x+c)^2 + y^2,$$

$$\therefore\; a^2 + 2cx + \frac{c^2x^2}{a^2} = x^2 + 2cx + c^2 + y^2,$$

$$\therefore\; y^2 = a^2 - c^2 - \frac{a^2 - c^2}{a^2}x^2, \quad \text{or } y^2 = \frac{a^2 - c^2}{a^2}(a^2 - x^2).$$

Now $SP + HP$ is greater than SH, or a is greater than c, therefore $a^2 - c^2$ is a positive quantity; hence, comparing the above with the equation to an ellipse

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

we see that the equations are identical, and consequently so are the curves which they represent, if $b^2 = a^2 - c^2$; therefore the required locus is an ellipse whose major axis is $2a$, and minor axis $2\sqrt{a^2 - c^2}$.

119. To find the polar equation to the ellipse, one of the foci being the pole.

Let the polar co-ordinates of any point P be $SP = r$, $\angle xSP = \theta$ (fig. 38); then

$$SP = e\,.\,PM = e\,(XS + SN),$$

$$\text{or } r = e\left\{\frac{a(1-e^2)}{e} + r\cos\theta\right\} \text{ (Art. 110)};$$

$$\therefore\; r(1 - e\cos\theta) = a(1 - e^2),$$

$$\text{or } r = \frac{a(1-e^2)}{1 - e\cos\theta}.$$

We have measured the angle θ from that part of the axis major which passes through the vertex furthest from

the pole; sometimes it is measured from the nearer vertex A, in which case if $ASP = \theta'$, putting $\pi - \theta'$ for θ we get

$$r = \frac{a(1-e^2)}{1+e\cos\theta'}.$$

Of course if the other focus H be taken for the pole, the formulæ will be exactly the same.

120. To find the polar equation to the ellipse, the centre being the pole.

Let $CP = r$, $\angle ACP = \theta$ (fig. 40), be the polar, and x, y, the rectangular co-ordinates of any point P; then $x = r\cos\theta$, $y = r\sin\theta$; therefore substituting these values in the equation

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

which, since $\frac{b^2}{a^2} = 1 - e^2$, may be written

$$y^2 + (1-e^2)x^2 = b^2, \text{ we get}$$

$$r^2(\sin^2\theta + \cos^2\theta - e^2\cos^2\theta) = b^2,$$

$$\text{or } r^2(1 - e^2\cos^2\theta) = b^2;$$

$$\therefore r = \frac{b}{\sqrt{1 - e^2\cos^2\theta}}.$$

In this formula it is indifferent from which vertex the angle θ is measured; it shews that of all lines drawn from the centre to the curve, the semi-axis major is the greatest, corresponding to $\theta = 0$; and the semi-axis minor the least, corresponding to $\theta = \frac{1}{2}\pi$.

To get the polar equation from the centre in terms of the semi-axes, we must substitute $r\cos\theta$ for x, and $r\sin\theta$ for y, in the equation $a^2y^2 + b^2x^2 = a^2b^2$ and the result is $r^2(a^2\sin^2\theta + b^2\cos^2\theta) = a^2b^2$.

121. To find the equation to the tangent of an ellipse at a given point.

As in former cases (Arts. 55 and 76), we shall regard the tangent as a secant which passes at first through two points of the curve, and then turns about the given point till the other point moves up to and coincides with it; so that if m be the tangent of the angle which it ultimately makes with the axis of x, and x', y', the co-ordinates of the given point, its equation will be

$$y - y' = m(x - x'), \text{ (Art. 20)}$$

where m is to be found in terms of x' and y'.

Let x'', y'', be the co-ordinates of a point in the curve near the proposed point, and α' the angle which the line joining them makes with the axis of x;

$$\text{then } \tan\alpha' = \frac{y' - y''}{x' - x''}.$$

But the two points being in the ellipse, their co-ordinates must satisfy its equation;

$$\therefore\ a^2y'^2 + b^2x'^2 = a^2b^2,\quad a^2y''^2 + b^2x''^2 = a^2b^2.$$

Subtracting the latter from the former, we get

$$a^2(y'^2 - y''^2) + b^2(x'^2 - x''^2) = 0,$$

$$\text{which gives } \tan\alpha' = \frac{y' - y''}{x' - x''} = -\frac{b^2}{a^2}\cdot\frac{x' + x''}{y' + y''}.$$

Now let the second point move up to and coincide with the first, then $x'' = x'$, $y'' = y'$, and the secant becomes a tangent at (x', y'); therefore, denoting by α the angle which the tangent makes with the axis of x, we get

$$m = \tan\alpha = -\frac{b^2x'}{a^2y'};$$

and consequently the equation to the tangent is

$$y - y' = -\frac{b^2 x'}{a^2 y'} \cdot (x - x'),$$

or, under another form, recollecting that $a^2 y'^2 + b^2 x'^2 = a^2 b^2$,

$$a^2 y y' + b^2 x x' = a^2 b^2;$$

in which x', y', are the co-ordinates of the point of contact, and x, y, co-ordinates of any point in the tangent line.

122. The formula $m = \tan a = -\frac{b^2 x'}{a^2 y'}$, since it does not change when x' and y' are replaced by $-x'$ and $-y'$, shews that if PC be produced to meet the ellipse in P' (fig. 39), the tangents at P and P' are parallel; as we might have foreseen from the symmetrical position of the ellipse, relative to the axes.

Also it proves that at the points B, B', for which $x' = 0$, $y' = \pm b$, $\tan a = 0$, or the tangents are parallel to AA'; and at A, A', for which $x' = \pm a$ and $y' = 0$, the tangents are perpendicular to AA'; and that for intermediate points, going from A to B, the angle PTx, which is always obtuse, continually increases till PT becomes parallel to AA' at B.

123. It is sometimes convenient to have the equation to the tangent expressed in terms of the angle which it makes with the major axis. The equation last written down (Art. 121) gives

$$y = mx + \frac{b^2}{y'};$$

but from $m = -\frac{b^2}{a^2}\frac{x'}{y'}$, we get $\left(\frac{am}{b}\right)^2 = \frac{b^2 x'^2}{a^2 y'^2}$

$$= \frac{a^2 b^2 - a^2 y'^2}{a^2 y'^2} = \frac{b^2}{y'^2} - 1, \text{ or } a^2 m^2 + b^2 = \frac{b^4}{y'^2};$$

$$\therefore y = mx \pm \sqrt{b^2 + m^2 a^2},$$

the lower sign referring to the point P'.

124. To find where the tangent meets the axis major, put the ordinate of the tangent $y = 0$ in the equation to the tangent; then $b^2 xx' = a^2 b^2$,

$$\therefore x = \frac{a^2}{x'}, \text{ or } CT = \frac{CA^2}{CN}.$$

As this result is independent of b, it will be the same for all ellipses constructed upon AA' as an axis; consequently, if NP meet the circle whose diameter is AA' in Q, the tangent to the circle at Q will pass through T.

Similarly, putting $x = 0$, to find where the tangent meets the axis minor, we get from the equation to the tangent

$$y = \frac{b^2}{y'}, \text{ or } Ct \, . \, PN = BC^2.$$

Subtracting CN or x' from the value just found for CT, we get the subtangent

$$NT = \frac{a^2}{x'} - x' = \frac{a^2 - x'^2}{x'}.$$

125. To find the equation to the normal at any point of an ellipse.

It will be of the form $y - y' = m'(x - x')$, x', y', being the co-ordinates of the proposed point; and since the normal is perpendicular to the tangent,

$$m' = -\frac{1}{m} = \frac{a^2 y'}{b^2 x'}; \text{ (Art. 121)}$$

therefore the equation to the normal is

$$y - y' = \frac{a^2 y'}{b^2 x'} \, . \, (x - x').$$

This equation, in the same way as for the tangent, may be expressed in terms of m'; for we get

$$y - m'x + m'\left(1 - \frac{y'}{m'x'}\right) x' = 0,$$

$$\text{or } y - m'x + m'\left(1 - \frac{b^2}{a^2}\right)x' = 0;$$

$$\text{but } \left(\frac{m'b}{a}\right)^2 = \frac{a^2y'^2}{b^2x'^2} = \frac{a^2}{x'^2} - 1,$$

$$\therefore \sqrt{a^2 + m'^2b^2} = \frac{a^2}{x'};$$

$$\therefore (y - m'x)\sqrt{a^2 + m'^2b^2} + m'(a^2 - b^2) = 0.$$

126. The normal at any point bisects the angle between the focal distances of that point.

First, to determine the point G, where the normal meets the axis major, make the ordinate of the normal $y = 0$, in the equation to the normal,

$$\therefore x = x'\left(1 - \frac{b^2}{a^2}\right), \quad \text{or } CG = e^2 . CN \text{ (fig. 41)};$$

$$\text{and } x' - x = \frac{b^2}{a^2}x', \text{ or the subnormal } GN = \frac{b^2}{a^2} . CN.$$

Hence $SG = SC + CG = ae + e^2x' = e(a + ex') = e.SP$,

and $HG = SC - CG = ae - e^2x' = e(a - ex') = e.HP$, (Art. 115)

$$\therefore \frac{SG}{HG} = \frac{SP}{HP},$$

and consequently the normal PG bisects the angle SPH (Euc. VI. 3). Also, drawing the tangent YZ at P, since $\angle GPY = GPZ$, each of them being a right angle, and $\angle GPS = GPH$; therefore $\angle SPY = HPZ$, or the focal distances make equal angles with the tangent on the same side of it; in other words the tangent bisects the exterior angle between the focal distances.

127. Drawing GL perpendicular to SP we get from similar triangles

$$SL = \frac{SG}{SP} . SN = e(ae + x);$$

$$\therefore PL = SP - SL = a + ex - e(ae + x) = a(1 - e^2),$$

which shews that if from G the foot of the normal at P we draw GL perpendicular to either focal distance, then $PL = \frac{1}{2}$ the latus rectum. Also

$$\tan SPY = \cot SPG = \frac{PL}{GL} = \frac{PL}{e.SP.\sin A'SP} = \frac{1 + e\cos A'SP}{e\sin A'SP},$$

which determines the angle at which the focal distance cuts the ellipse.

If we call $ST = r$, $AST = \theta$, $ASP = \alpha$, and $SPT = \phi$, (fig. 30) we have

$$\frac{SP}{r} = \frac{\sin STP}{\sin SPT} = \frac{\sin(\phi + \alpha - \theta)}{\sin\phi} = \cos(\alpha - \theta) + \sin(\alpha - \theta)\cot\phi$$

$$= \cos(\alpha - \theta) + \sin(\alpha - \theta) \cdot \frac{e\sin\alpha}{1 + e\cos\alpha} = \{\cos(\alpha - \theta) + e\cos\theta\}\frac{SP}{a(1 - e^2)};$$

$$\therefore r = \frac{a(1 - e^2)}{\cos(\alpha - \theta) + e\cos\theta}$$

the polar equation to the tangent of the ellipse, which is sometimes of use.

128. These properties furnish a simple method of drawing a tangent to an ellipse through a given point.

First, let the point be in the ellipse as P (fig. 42).

Join SP, HP, and produce SP to K, making $SK = 2AC$; join HK and draw PZ perpendicular to it, then PZ is a tangent at P.

For in the triangles PHZ, PKZ, $PK = 2AC - SP = PH$, PZ is common, and the angles at Z are right angles,

$$\therefore \angle HPZ = KPZ = SPY,$$

and consequently PZ is a tangent at P.

Next, let the point be without the ellipse.

With S the focus furthest from T the given external

point, as centre, (fig. 43), and radius $= 2AC$, describe a circle KK'; and with centre T, and radius equal to TH the distance of T from the other focus, describe a circle cutting the former in K, K'; join SK meeting the ellipse in P, and join TP; then TP is a tangent at P; for in the triangles TPK, TPH, $PK = PH$, $TK = TH$, and TP is common, therefore TP bisects the exterior angle HPK, and is a tangent at P; similarly, if SK' be joined, it will meet the ellipse in P' a second point of contact.

As long as T is exterior to the ellipse, the circles must intersect. For if $ST > 2AC$, T and H fall on opposite sides of the circumference KK'; but if ST be less than $2AC$, join ST and produce it to meet KK' in T'; then $ST + TH > ST'$ (Art. 117), therefore $TH > TT'$; and therefore in both cases the circles intersect. If KK' be joined, it is evident that $\angle TKP = TK'P'$; therefore $THP = THP'$; or the tangents drawn from an external point subtend equal angles at either focus.

129. This problem may be also solved by means of the equation to the tangent. Let h, k be the co-ordinates of the given external point, and x', y', those of the unknown point of contact; then since x' and y' must satisfy both the equation to the tangent and that to the ellipse, we have, to determine them,

$$b^2hx' + a^2ky' = a^2b^2,\ b^2x'^2 + a^2y'^2 = a^2b^2.$$

It is evident that x' and y' will each have two values, therefore there will be two points of contact; and if we construct the line represented by the former of these equations regarding x' and y' as the variables, it will intersect the ellipse in the points of contact.

Hence the chord joining the points of contact of two tangents drawn from a point (h, k), has for its equation

$$b^2hx + a^2ky = a^2b^2,$$

and it meets the axes of the ellipse in points for which $x = \dfrac{a^2}{h}$,

$y = \frac{b^2}{k}$; which values shew that if two tangents be drawn from any point in a line parallel to either axis, the chord joining the points of contact will pass through a fixed point in the other axis, and conversely.

To find the angle α between the tangents that intersect in a given point we have, if m, m', be the tangents of the angles which the touching lines make with the axis,

$$(1 + mm')^2 \tan^2 \alpha = (m - m')^2 = (m + m')^2 - 4mm',$$

$$\text{or } \left(1 + \frac{k^2 - b^2}{h^2 - a^2}\right)^2 \tan^2 \alpha = \left(\frac{2hk}{h^2 - a^2}\right)^2 - 4\frac{k^2 - b^2}{h^2 - a^2},$$

since m, m', are the roots of $(h^2 - a^2)m^2 - 2hkm + k^2 - b^2 = 0$ (Art. 123). If α be invariable, then

$$(h^2 + k^2 - a^2 - b^2)^2 \tan^2 \alpha = 4(h^2 b^2 + k^2 a^2 - a^2 b^2)$$

is the equation to the locus of the intersection of two tangents to an ellipse that include a constant angle.

130. The locus of the extremities of the perpendiculars dropped from the foci upon the tangent to an ellipse, is the circumference of the circle whose diameter is the axis major. Produce any focal distance SP (fig. 42) to K, so that $SK = 2AC$, join HK, and draw PZ perpendicular to it; then PZ bisects HK, and is a tangent at P (Art. 128). Join CZ, then since SH is bisected in C and HK in Z, CZ is parallel to SK and equal to $\frac{1}{2} SK = AC$. Also, drawing SY, CQ, parallel to HZ, CQ bisects YZ perpendicularly, and therefore $CY = CZ = AC$. Hence the intersections of every tangent with the perpendiculars upon it from the foci, are at a constant distance AC from the centre of the ellipse; or are situated in the circumference of the circle whose diameter is the axis major.

131. Since C is the centre of the circle which is the locus of Y and Z, and SYZ is a right angle, and therefore

in a semicircle, if YS and ZC be produced to meet in H', this will be a point in the circle; and from the equal triangles CZH, $CH'S$, $SH' = HZ$;

$$\therefore\ SY \times HZ = SY \times SH' = AS \times A'S = BC^2 \text{ (Art. 109).}$$

Also, since $\dfrac{SY}{SP} = \dfrac{HZ}{HP}$, or $\dfrac{SY}{HZ} = \dfrac{SP}{HP}$ (Art. 126);

multiplying this equation by the preceding, we get

$$SY^2 = BC^2 \times \frac{SP}{HP};$$

or, if SY be denoted by p and SP by r, and consequently HP by $2a - r$, we have, between the radius vector of any point and the perpendicular on the tangent at that point from the focus, the relation

$$p^2 = b^2 \cdot \frac{r}{2a - r}.$$

Draw CE parallel to PY, then PC is a parallelogram; therefore $PE = CZ = AC$; which shews that the portion of any focal distance cut off by the diameter parallel to the tangent at its extremity, is invariable.

132. The preceding results may be also readily obtained by means of the equation to the tangent (Art. 123). For the equations to PT and HZ are, respectively,

$$y = mx + \sqrt{m^2 a^2 + b^2},$$

$$y = -\frac{1}{m}(x - \sqrt{a^2 - b^2}),$$

between which if we eliminate m we shall obtain the equation to the locus of their intersection.

These equations may be written

$$y - mx = \sqrt{m^2 a^2 + b^2}, \quad x + my = \sqrt{a^2 - b^2};$$

and adding their squares, the result is

$$(x^2 + y^2)(m^2 + 1) = a^2(m^2 + 1), \quad \text{or } x^2 + y^2 = a^2,$$

the equation to the locus of Z.

Again, since HZ is the perpendicular dropped from a point whose co-ordinates are $x' = \sqrt{a^2 - b^2}$, $y' = 0$, upon a line whose equation is $y = mx + \sqrt{m^2a^2 + b^2}$, we have (Art. 28),

$$HZ = \frac{-m\sqrt{a^2 - b^2} - \sqrt{m^2a^2 + b^2}}{\sqrt{1 + m^2}};$$

$$\text{similarly, } SY = \frac{m\sqrt{a^2 - b^2} - \sqrt{m^2a^2 + b^2}}{\sqrt{1 + m^2}},$$

$$\therefore\ HZ \times SY = \frac{m^2a^2 + b^2 - m^2(a^2 - b^2)}{m^2 + 1} = b^2.$$

133. Draw HI parallel to YZ (fig. 42) and let $\angle SPY = \alpha$;

$$\text{then } \tan STP = \tan SHI = \frac{SI}{IH} = \frac{r \sin\alpha - (2a - r)\sin\alpha}{r\cos\alpha + (2a - r)\cos\alpha},$$

$$\text{or } \tan STP = \frac{2(r - a)}{2a} \tan\alpha = \left(\frac{SP}{AC} - 1\right) \tan SPY;$$

a result which is sometimes of use.

134. The tangents at the extremities of any focal chord intersect in the directrix; and the line joining their intersection with the focus is perpendicular to the chord.

Let h, k, be the co-ordinates of the intersection of the tangents; then the equation to the chord joining the points of contact is (Art. 129)

$$b^2hx + a^2ky = a^2b^2,$$

and since it passes through the focus, when $x = ae$, we must have $y = 0$; therefore $h = \frac{a}{e}$, the equation to the directrix.

Also the equation to a perpendicular to this chord through the focus is

$$y = \frac{k}{h}\frac{a^2}{b^2}(x - ae); \text{ and when } x = \frac{a}{e},\ y = \frac{ka}{he} = k,$$

therefore it passes through the intersection of the tangents.

Also, if $HP = r$, $HP' = r'$, $HZ = c$, $AHP = \theta$, (fig. 41),

$$\text{then } \tan PZP' = c \cdot \frac{r + r'}{c^2 - rr'};$$

$$\text{but } \frac{1}{r} + \frac{1}{r'} = \frac{2}{a(1 - e^2)},\quad rr' = \frac{a^2(1 - e^2)^2}{1 - e^2\cos^2\theta},\quad c = \frac{a(1 - e^2)}{e\sin\theta},$$

$$\text{therefore, substituting, } \tan PZP' = \frac{2e\sin\theta}{1 - e^2}.$$

135. If a perpendicular be dropped from the centre upon the tangent at any point of an ellipse, making an angle ϕ with the axis major, its length $= a\sqrt{1 - e^2\sin^2\phi}$.

Let CQ, HZ (fig. 44), be perpendiculars dropped upon PT the tangent at any point P, from the centre and focus; join CZ and let $\angle TCQ = \phi$;

$$\text{then } CZ = a, \text{ and } QZ = CH\sin\phi = ae\sin\phi;$$

$$\therefore\ CQ^2 = a^2 - a^2e^2\sin^2\phi, \quad \text{or } CQ = a\sqrt{1 - e^2\sin^2\phi}.$$

Hence the polar equation to the locus of the foot of the perpendicular dropped from the centre upon the tangent to an ellipse, is $r = a\sqrt{1 - e^2\sin^2\phi}$.

Also if $P'T'$ be another tangent to the ellipse at right angles to the former, and CQ' perpendicular to it; then $Q'CT' = \frac{1}{2}\pi - \phi$; and therefore $CQ'^2 = a^2 - a^2e^2\cos^2\phi$.

$$\text{Hence } CW^2 = CQ^2 + CQ'^2 = a^2 + a^2(1 - e^2) = a^2 + b^2,$$

and therefore the locus of the intersection of two tangents to an ellipse at right angles to one another, is a circle whose centre is C and radius equal to $\sqrt{a^2 + b^2}$.

These results may be also obtained in the following manner. The equation to PT being

$$y - mx = \sqrt{m^2a^2 + b^2} \quad (1),$$

the equation to CQ is $y = -\frac{1}{m}x$; and eliminating m between these equations, i.e. substituting in the former $-\frac{x}{y}$ for m, we get for the locus of Q the equation $y^2 + x^2 = \sqrt{a^2x^2 + b^2y^2}$. Again the equation to $P'W$ is $my + x = \sqrt{a^2 + m^2b^2}$; and adding the square of this to the square of (1) and dividing by $m^2 + 1$, we get $x^2 + y^2 = a^2 + b^2$, for the equation to the locus of W.

136. It is sometimes convenient to have the length of the normal to an ellipse, and also the co-ordinates of the point where it meets the curve expressed in terms of the inclination of the normal to the major axis.

From similar triangles (fig. 44), we get

$$\frac{PG}{PN} = \frac{Ct}{CQ},$$

$$\therefore \; PG = \frac{Ct \, . \, PN}{CQ} = \frac{b^2}{a\sqrt{1 - e^2 \sin^2 \phi}} \text{ (Art. 124 and 135).}$$

$$\text{Also } PN = \frac{b^2 \sin \phi}{a\sqrt{1 - e^2 \sin^2 \phi}},$$

$$CN = \frac{a^2}{b^2} GN = \frac{a \cos \phi}{\sqrt{1 - e^2 \sin^2 \phi}}, \quad CP = \frac{\sqrt{a^4 \cos^2 \phi + b^4 \sin^2 \phi}}{a\sqrt{1 - e^2 \sin^2 \phi}}.$$

137. All chords of an ellipse which subtend a right angle at a given point of the curve, intersect one another in the normal at that point.

Take the given point for origin, and the normal and tangent at that point for axes of x and y; then the equation

to the curve (which includes every species of conic section) will be

$$ay^2 + bxy + cx^2 + dy + ex + f = 0;$$

and since the axis of y is a tangent at the origin, if $x = 0$, the values of y become each $= 0$,

$$\therefore\ d = 0,\ f = 0, \text{ and the equation is reduced to}$$

$$ay^2 + bxy + cx^2 + ex = 0.$$

Let $y = m(x - h)$ be the equation to any chord meeting the normal at a distance h from the origin; then for the points of intersection with the curve,

$$am^2(x-h)^2 + bmx(x-h) + cx^2 + ex = 0,$$

of which equation, if x_1, x_2, be the roots,

$$x_1x_2 = \frac{am^2h^2}{am^2 + bm + c}.$$

Also eliminating x,

$$ay^2 + by\left(\frac{y}{m} + h\right) + c\left(\frac{y}{m} + h\right)^2 + e\left(\frac{y}{m} + h\right) = 0;$$

and if y_1, y_2, be the values of y,

$$y_1y_2 = \frac{cm^2h^2 + em^2h}{am^2 + bm + c};$$

if now the chord subtend a right angle at origin, the lines joining its extremities with that point, whose equations are $y = \frac{y_1}{x_1}x$, $y = \frac{y_2}{x_2}x$, will be perpendicular to one another;

$$\therefore\ \frac{y_1}{x_1} = -\frac{x_2}{y_2}, \quad \text{or } x_1x_2 + y_1y_2 = 0;$$

$$\therefore\ ah^2 + ch^2 + eh = 0, \text{ which gives } h = -\frac{e}{a+c},$$

a value constant for all values of m; hence all such chords pass through a fixed point in the normal.

138. To determine the intersection of a straight line with an ellipse.

Let $y = mx + c$ be the equation to any straight line; if this line intersect an ellipse whose equation is

$$a^2y^2 + b^2x^2 = a^2b^2,$$

and if x', y', be the co-ordinates of a common point, we have

$$y' = mx' + c, \quad a^2y'^2 + b^2x'^2 = a^2b^2;$$

therefore, substituting $\frac{1}{m}(y' - c)$ for x' in the latter, we get

$$(a^2m^2 + b^2)y'^2 - 2b^2cy' + (c^2 - m^2a^2)b^2 = 0,$$

the roots of which are the ordinates of the points where the straight line meets the curve, and the abscissæ are known from $x' = \frac{1}{m}(y' - c)$. Hence if the roots be real, the straight line will cut the ellipse in two points, and it cannot cut the ellipse in more than two points; if the roots are imaginary, the line falls entirely without the ellipse.

Obs. If the roots be equal, the points of section coincide, and the line is then a tangent; and we have

$$(a^2m^2 + b^2)(c^2 - m^2a^2) = b^2c^2, \quad \text{or } c^2 = b^2 + m^2a^2;$$

$$\therefore y = mx \pm \sqrt{b^2 + m^2a^2}$$

is the equation to the tangent to an ellipse in terms of the angle which it makes with the major axis, agreeably to Art. 123.

139. To find the locus of the middle points of a system of parallel chords.

Let the chords be parallel to a line CD (fig. 45) through the centre, whose equation is $y = mx$; then the equation to

any one of them QQ' is $y = mx + c$; and the values of QM, $Q'M'$, are the roots of the equation,

$$y^2 - \frac{2b^2c}{a^2m^2 + b^2}y + \frac{b^2(c^2 - m^2a^2)}{a^2m^2 + b^2} = 0,$$

obtained by eliminating x between the equations to the chord and the ellipse, as in Art. 138.

If therefore V be the middle point of QQ', and $CN = X$, $VN = Y$, its co-ordinates, so that $Y = mX + c$,

$$Y = \tfrac{1}{2}(QM + Q'M') = \frac{b^2c}{m^2a^2 + b^2},$$

$$X = \frac{1}{m}(Y - c) = -\frac{ma^2c}{m^2a^2 + b^2};$$

therefore, dividing one result by the other in order to eliminate the quantity c which particularizes the chord, we get

$$Y = -\frac{b^2}{ma^2} \cdot X,$$

a relation between the co-ordinates of the middle point of any chord, and therefore the equation to its locus, which is consequently a straight line PP' passing through the origin.

The straight line which passes through the middle points of a system of parallel chords is called a Diameter.

Hence all diameters of an ellipse pass through its centre; and, conversely, every line through the centre may be con sidered as a diameter.

Hence denoting the equations to any chord QQ', and to the diameter PP' which bisects it, by $y = mx + c$, $y = m'x$, respectively,

$$\text{we have } m' = -\frac{b^2}{ma^2}, \text{ or } mm' = -\frac{b^2}{a^2},$$

a simple relation, by means of which the equation of a diameter may always be deduced from that of any chord which it bisects, or *vice versa*.

140. If a diameter PP' bisect the chords parallel to a given diameter DD', then likewise the diameter DD' will bisect the chords parallel to PP'.

Let $y = mx$ be the equation to the given diameter

DD' (fig. 45); then (Art. 139) $y = -\dfrac{b^2}{ma^2}x$, or

$$y = m'x, \text{ if } m' = -\frac{b^2}{ma^2},$$

is the equation to the diameter bisecting the chords parallel to DD', which diameter by supposition is PP'. Now let $y = nx$ be the equation to the diameter bisecting the chords parallel to PP'; then $n = -\dfrac{b^2}{m'a^2}$, or $n = m$; therefore DD' is the diameter bisecting the chords parallel to PP'.

Hence two diameters, whose equations $y = mx$, $y = m'x$, are so related that $mm' = -\dfrac{b^2}{a^2}$, have the property that each bisects the chords parallel to the other.

Two diameters, which thus mutually bisect the chords parallel to one another, are called Conjugate Diameters, or rather those portions of them PP', DD', which fall within the ellipse are usually called a pair of conjugate diameters.

141. If PT be the tangent at P, and x', y', the coordinates of P, then the equation to PT is (Art. 121)

$$y - y' = -\frac{b^2x'}{a^2y'}(x - x');$$

but the equation to CP is $y = \dfrac{y'}{x'}x$, and therefore the equation to CD, the diameter conjugate to CP, is $y = -\dfrac{b^2x'}{a^2y'}x$, which represents a line parallel to PT.

Hence the tangent at the extremity of any diameter, is parallel to the corresponding conjugate diameter; and if tangents be applied at the extremities of a pair of conjugate diameters, they will form a parallelogram circumscribing the ellipse (Art. 122).

This result might have been foreseen; for DD' being parallel to all chords bisected by PP' is parallel to that which is situated indefinitely near to P and which ultimately coincides in direction with the tangent at P when the two points in which it meets the curve become coincident in P.

142. Having given the co-ordinates of the extremity of any diameter, to find those of the extremity of the diameter conjugate to it.

Let x', y', be the co-ordinates of the point P (fig. 46),

$$\text{then } y = \frac{y'}{x'}x \text{ is the equation to } CP,$$

$$\text{and } \therefore\ y = -\frac{b^2x'}{a^2y'}x \text{ the equation to } CD.$$

To determine the co-ordinates of D, we must combine the equation to CD with the equation to the ellipse $a^2y^2+b^2x^2=a^2b^2$, which gives, eliminating y by the substitution $-\frac{b^2x'}{a^2y'}x$,

$$\frac{b^4}{a^2}\cdot\frac{x'^2}{y'^2}\cdot x^2 + b^2x^2 = a^2b^2,$$

$$\text{or } x^2(b^2x'^2 + a^2y'^2) = a^4y'^2, \text{ or } x^2a^2b^2 = a^4y'^2;$$

$$\therefore\ x^2 = \frac{a^2y'^2}{b^2}, \text{ or } -x = CM = \frac{ay'}{b},$$

$$\text{and } y = -\frac{b^2}{a^2}\,\frac{x'}{y'}\,x = DM = \frac{bx'}{a};$$

the other pair of values of x and y having reference to D'.

Hence, if we suppose the ordinates NP, MD, produced to

meet the circle on the major axis in p, d, $Np = \frac{a}{b}y' = CM$, and $Md = CN$, and consequently the angle pCd is a right angle.

143. The sum of the squares of any two semi-conjugate diameters is equal to the sum of the squares of the semi-axes.

$$CP^2 = CN^2 + NP^2 = x'^2 + y'^2, \text{ (fig. 46)}$$

$$CD^2 = CM^2 + MD^2 = \frac{a^2 y'^2}{b^2} + \frac{b^2 x'^2}{a^2}; \text{ (Art. 142)}$$

$$\therefore\ CP^2 + CD^2 = (a^2 + b^2)\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right) = a^2 + b^2,$$

$$\text{since } \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1.$$

144. All parallelograms whose sides touch an ellipse at the extremities of a pair of conjugate diameters, are equal to one another.

Draw CQ (fig. 46) perpendicular to the tangent at P.

Then the area of the whole parallelogram

$$\begin{aligned} &= \text{four times the parallelogram } DP \\ &= 4CD \,.\, CQ = 4CD \,.\, CT \sin CTQ \\ &= 4CT \,.\, CD \sin DCM = 4CT \,.\, DM \\ &= 4 \,.\, \frac{a^2}{x'} \,.\, \frac{bx'}{a} = 4ab \text{ (Arts. 124 \& 142).} \end{aligned}$$

145. If we denote CP, CD, by a', b', and $\angle DCP$ by γ, and draw PF perpendicular to DC produced, we have $CQ = PF = a' \sin\gamma$,

$$\therefore\ a'b' \sin\gamma = CD \,.\, PF = ab.$$

This equation, together with $a'^2 + b'^2 = a^2 + b^2$, determines the magnitudes $2a'$, $2b'$, of two conjugate diameters that in-

clude a given angle γ; and their position is known from the equation

$$CQ = a' \sin\gamma = a\sqrt{1 - e^2 \sin^2\phi} \text{ (Art. 135)},$$

$$\text{where } \phi = QCN = \frac{\pi}{2} - DCM,$$

$$\text{which gives } \sin\phi = \cos DCM = \frac{a}{b'} \sqrt{\frac{b'^2 - b^2}{a^2 - b^2}}.$$

If $a' = b'$, then

$$a'^2 = \tfrac{1}{2}(a^2 + b^2), \ \sin\gamma = \frac{ab}{a'^2} = \frac{2ab}{a^2 + b^2}, \text{ or } \tan\frac{\gamma}{2} = \frac{a}{b};$$

$$\text{and } \sin\phi = \frac{a}{\sqrt{a^2 + b^2}}; \text{ or } \cot\phi = \tan DCM = \frac{b}{a},$$

as might have been foreseen; for the equal conjugate diameters being symmetrically situated with respect to the major axis,

$$mm' = -\frac{b^2}{a^2} \text{ gives } \tan^2 DCM = \frac{b^2}{a^2}.$$

Hence the equal conjugate diameters of an ellipse are parallel to the chords joining its vertices; and the angle between them is $2\tan^{-1}\frac{a}{b}$. In this case, $\sin\gamma$ receives its least value; for it is least when $2a'b' = a^2 + b^2 - (a' - b')^2$ is greatest; i. e., when $a' = b'$. Hence the obtuse angle between two conjugate diameters varies from $\frac{\pi}{2}$ the angle between the axes, to $2\tan^{-1}\frac{a}{b}$ the angle between the equal conjugate diameters.

146. If we denote CQ or PF by p, we have

$$p^2 = \frac{a^2b^2}{CD^2} = \frac{a^2b^2}{a^2 + b^2 - a'^2}, \text{ (Art. 143)}$$

a relation between the central distance a' of a point, and the

perpendicular p upon the tangent at that point, let fall from the centre.

147. The rectangle contained by the focal distances of any point is equal to the square of the corresponding semi-conjugate diameter.

$$CD^2 = a^2 + b^2 - CP^2 \text{ (fig. 40) (Art. 143)}$$

$$\text{but } CP^2 = x^2 + y^2 = x^2 + b^2 - \frac{b^2}{a^2}x^2 = b^2 + e^2x^2,$$

$$\therefore CD^2 = a^2 - e^2x^2 = (a + ex).(a - ex)$$
$$= SP.HP \text{ (Art. 115)}.$$

148. To find the equation to the ellipse referred to the system of oblique axes formed by any pair of conjugate diameters.

Let x and y be the rectangular co-ordinates of any point Q in the ellipse referred to its centre and axes; then the relation between them is

$$a^2y^2 + b^2x^2 = a^2b^2.$$

Let the conjugate diameters CP, CD (fig. 47), be the new axes of x' and y', inclined to the axis of x at angles $PCA = \alpha$, $DCA = \beta$, and $CV = x'$, $QV = y'$, the new co-ordinates of Q; then since the origin remains unaltered, the formulæ for passing from the rectangular to the oblique axes are (Art. 42),

$$x = x' \cos\alpha + y' \cos\beta, \quad y = x' \sin\alpha + y' \sin\beta.$$

Hence, substituting in the above equation and reducing,

$$(a^2 \sin^2\alpha + b^2 \cos^2\alpha)\, x'^2 + (a^2 \sin^2\beta + b^2 \cos^2\beta)\, y'^2$$
$$+ 2\,(a^2 \sin\alpha \sin\beta + b^2 \cos\alpha \cos\beta)\, x'y' = a^2b^2.$$

$$\text{But } \tan\alpha \tan\beta = -\frac{b^2}{a^2} \text{ (Art. 140)};$$

$$\therefore a^2 \sin\alpha \sin\beta + b^2 \cos\alpha \cos\beta = 0,$$

and the term involving $x'y'$ disappears.

Also if $CP = a'$, $CD = b'$, we have (Art. 120),

$$a'^2 (a^2 \sin^2 \alpha + b^2 \cos^2 \alpha) = a^2 b^2,$$
$$b'^2 (a^2 \sin^2 \beta + b^2 \cos^2 \beta) = a^2 b^2;$$

therefore, substituting and dividing by $a^2 b^2$, we get for the required equation,

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} = 1, \text{ or } y'^2 = \frac{b'^2}{a'^2}(a'^2 - x'^2);$$

or, in a geometrical form, supposing PC produced to meet the ellipse in G, so that $PV = a' - x'$, and $VG = a' + x'$,

$$QV^2 = \frac{CD^2}{CP^2} . PV . VG.$$

149. This equation, which, suppressing the accents of the variables, is $\dfrac{x^2}{a'^2} + \dfrac{y^2}{b'^2} = 1$, being of precisely the same form as that relative to the axes, it follows that all properties which do not depend upon the inclination of the co-ordinates, will be common to the Axes of the ellipse and to its conjugate diameters.

Hence, x', y', being the co-ordinates of any point Q referred to the conjugate diameters CP, CD (fig. 47), the equation to the tangent at that point will be (Art. 121)

$$a'^2 yy' + b'^2 xx' = a'^2 b'^2;$$

and if the tangent meet the co-ordinate axes in T, t, we shall have, as before, (Art. 124),

$$CT = \frac{a'^2}{x'} = \frac{CP^2}{CV}, \quad Ct = \frac{CD^2}{QV}.$$

150. Also, if we wish to draw a tangent through an external point Q (fig. 48), whose co-ordinates are h and k, we shall have, to find the points of contact, the equations

$$a'^2 y'^2 + b'^2 x'^2 = a'^2 b'^2,$$
$$a'^2 ky' + b'^2 hx' = a'^2 b'^2;$$

the latter (considering x' and y' as the variables) being the equation to the chord joining the two points of contact.

And those points, as in preceding similar cases, may be determined, by constructing this line; that is, by taking $CT = \frac{a'^2}{h}$, $CR = \frac{b'^2}{k}$, and joining RT, which will cut the ellipse in the two required points.

Since the distance CT is independent of k, if through Q we draw a line parallel to CD, and from any other point in this line we draw a pair of tangents to the ellipse, the secant passing through the new points of contact, will cut the diameter CP in T, as this point only alters when h alters. Hence if from the several points of any straight line, pairs of tangents be drawn to an ellipse, the straight lines which join the corresponding points of contact will all pass through the same point; and conversely if through any point (h, k) we draw different chords, and apply two tangents at the extremities of each, the locus of the intersection of the tangents will be a straight line having for equation $a'^2ky + b'^2hx = a'^2b'^2$. If the line be the directrix, then as we have seen (Art. 134), the fixed point will be the focus.

151. The tangents at the extremities of any chord will intersect in the diameter of which the chord is an ordinate.

For, taking that diameter and its conjugate as the axes of x and y, the equation to the tangent will be

$$b'^2xx' \pm a'^2yy' = a'^2b'^2,$$

where the upper or lower sign is to be used, according as we consider the point $Q(x', y')$ (fig. 47), or the other extremity of the chord Q' whose co-ordinates are x', $-y'$, to be the point of contact; and in both cases when $y = 0$, $x = \frac{a'^2}{x'}$; therefore the tangents meet the axis of x in the same point T.

152. If from the extremities of any diameter, two chords be drawn to any point in an ellipse, and one of them be

parallel to a diameter, the other will be parallel to the conjugate diameter.

Join any point I (fig. 47) with the extremities of any diameter PG; and let x', y', be co-ordinates of P referred to the Axes of the ellipse, and consequently $-x'$, $-y'$, those of G, and x, y, those of I; then if m, m', be the tangents of the angles which IP, IG, make respectively with the axis of x,

$$m = \frac{y - y'}{x - x'}, \quad m' = \frac{y + y'}{x + x'}, \quad \therefore \ mm' = \frac{y^2 - y'^2}{x^2 - x'^2};$$

$$\text{but } \ a^2y^2 + b^2x^2 = a^2b^2, \quad a^2y'^2 + b^2x'^2 = a^2b^2,$$

$$\therefore, \text{ subtracting, } y^2 - y'^2 = -\frac{b^2}{a^2}(x^2 - x'^2), \quad \therefore \ mm' = -\frac{b^2}{a^2};$$

which shews (Art. 140) that if GI be parallel to a diameter, PI will be parallel to the conjugate diameter. Chords joining any point in the ellipse with the extremities of a diameter, are called supplemental chords of that diameter.

Conversely, if two chords, whose equations referred to the Axes are $y = mx + c$, $y = m'x + c'$, satisfy the condition $mm' = -\frac{b^2}{a^2}$, then, whether they both pass through the same point in the ellipse, or through the extremities of a diameter, they are supplemental to one another.

153. The angles between the supplemental chords of any diameter whatever are the same as those between the supplemental chords of the major axis.

For if from the extremities of the major axis A, A', lines be drawn parallel to PI, GI, as their equations will satisfy the condition $mm' = -\frac{b^2}{a^2}$, they will intersect in a point K in the ellipse, and the angle $AKA' = PIG$; that is, no angle can be contained by the chords of PG, but chords relative to AA' can be drawn, containing an equal angle. Hence the angle between the supplemental chords of every diameter

will be greater than a right angle and less than $2\tan^{-1}\frac{a}{b}$, these being the limits of the angle contained by supplemental chords relative to the axis major.

For let x, y, be co-ordinates of the point K referred to the axes, then $\tan KAx = \dfrac{y}{x-a}$, $\tan KA'x = \dfrac{y}{x+a}$,

$$\therefore \tan AKA' = \frac{2ay}{x^2 - a^2 + y^2} = \frac{2ay}{y^2\left(1 - \dfrac{a^2}{b^2}\right)} = -\frac{2ab^2}{y(a^2 - b^2)}.$$

Hence $\angle AKA'$ is always obtuse; and it is least, viz. a right angle when $y = 0$; and greatest and $= 2\tan^{-1}\frac{a}{b}$, when $y = b$.

154. Hence we can readily construct two conjugate diameters containing a given angle γ. Upon the major axis of the ellipse describe a segment of a circle (fig. 78) containing the given angle γ; then because γ lies between $\frac{1}{2}\pi$ and $2\tan^{-1}\frac{a}{b}$, there is a pair of supplemental chords which contain this angle, and therefore the circle must intersect the ellipse in one, and therefore in two points K, K'; and if AK, $A'K$ be joined and CD, CP, be drawn parallel to them, then CP, CD, are conjugate diameters, and they include $\angle DCP = AKA' = \gamma$. If the given angle be $2\tan^{-1}\frac{a}{b}$, the circle will touch the ellipse at B and only one system of conjugate diameters will be determined, viz. the equal ones; if the given angle be a right angle the chords $A'K$, AK will coincide with the tangent at A and with AA', and the axes of the ellipse will be determined.

155. If from any point within or without an ellipse two lines be drawn parallel to two given straight lines to meet the curve, the rectangles of the segments will be to one another in an invariable ratio.

Let O (fig. 49) be the given point with co-ordinates h, k. Then taking O for the pole and measuring θ from a line parallel to the axis major the polar equation to the ellipse will be (Art. 14)

$$a^2(r\sin\theta + k)^2 + b^2(r\cos\theta + h)^2 = a^2b^2,$$

which is of the form $r^2 + Mr - N = 0$, where

$$N = \frac{a^2b^2 - a^2k^2 - b^2h^2}{a^2\sin^2\theta + b^2\cos^2\theta} = r'r'',$$

if these be the two values of r.

Now let Pp, Qq, be drawn parallel to CP', CQ', which make given angles α, β, with Cx; then

$$PO \times Op : QO \times Oq :: a^2\sin^2\beta + b^2\cos^2\beta : a^2\sin^2\alpha + b^2\cos^2\alpha$$
$$:: CP'^2 : CQ'^2 \text{ (Art. 120)},$$

a ratio independent of the position of the point O.

Hence if we suppose Pp, Qq, to move parallel to themselves till they become tangents to the ellipse at points P and Q respectively, and intersect in a point O without the curve, we have

$$OP : OQ :: CP' : CQ'.$$

156. To find the area of the ellipse.

Let $APQRTA'$ (fig. 50) be any polygon inscribed in the ellipse, and let the ordinates PN, QM, &c. be produced to meet the circle on the major axis in p, q, r, &c. and join Ap, pq, &c.

Then area of trapezium $PNMQ = \frac{1}{2}(PN + QM) \,.\, NM$

$$= \frac{1}{2}\frac{b}{a}(pN + qM)\,.\,NM = \frac{b}{a}\,.\,\text{trapezium } pNMq,$$

$$\text{or } \frac{\text{area of trapezium } PM}{\text{area of trapezium } pM} = \frac{b}{a};$$

and since the same ratio exists between every two corresponding trapeziums,

$$\frac{\text{area of polygon } APQA'}{\text{area of polygon } ApqA'} = \frac{b}{a};$$

and this is true however much the number of the sides of the polygons be increased; therefore, supposing the number to be infinite, in which case the ratio of the polygons becomes that of the semi-ellipse and semicircle,

$$\frac{\text{area of semi-ellipse}}{\text{area of semicircle}} = \frac{b}{a};$$

$$\therefore \text{ area of ellipse} = \frac{b}{a}\pi a^2 = \pi a b.$$

Likewise if K (fig. 51) be any point in the axis, and QPN be an ordinate to the circle and ellipse, then

$$\text{elliptic area } ANP = \frac{b}{a}.\text{circular area } ANQ,$$

$$\text{and triangle } PKN = \frac{b}{a}.\text{triangle } QKN;$$

therefore, subtracting,

$$\text{the elliptic sectorial area } AKP = \frac{b}{a}.\text{circular area } AKQ.$$

Also, if a', b', be two semi-conjugate diameters and γ the angle between them, the area of the ellipse $= \pi a' b' \sin\gamma$ (Art. 145).

And the area of the sector bounded by the semi-diameters

$$= \tfrac{1}{4}\pi a' b' \sin\gamma.$$

SECTION VIII.

ON THE HYPERBOLA.

Various Forms of the Equation to the Hyperbola.

157. To find the equation to the Hyperbola.

The hyperbola is the locus of a point, whose distance from a given point is always greater than its distance from a given fixed line, in a constant ratio.

Let KK' (fig. 52) be the given fixed line, and S the given point, from which draw SX perpendicular to KK'. Let P be a point in the hyperbola on either side of KK'; and from P draw PM perpendicular to KK', and join SP, and let the constant ratio of SP to PM be $e : 1$, e being greater than 1. Divide the given distance SX in A so that $SA = e.AX$, then A is a point in the curve; and assuming $AS = p$, we have $AX = \dfrac{p}{e}$. Through A draw Ay parallel to KK', and take A for the origin, and Ax, Ay, for the co-ordinate axes, and let $AN = x$, $NP = y$, be the co-ordinates of P; then

$$SP^2 = e^2.PM^2,$$

$$\text{or } SN^2 + NP^2 = e^2.NX^2,$$

$$\text{or } (x - p)^2 + y^2 = e^2.\left(\frac{p}{e} + x\right)^2 = (p + ex)^2;$$

$$\therefore\ y^2 = 2p(e+1)x + (e^2 - 1)x^2,$$

$$\text{or } y^2 = (e^2 - 1)\left(\frac{2p}{e-1}x + x^2\right);$$

or, if we replace the known quantity $\dfrac{p}{e-1}$ by a,

$$y^2 = (e^2 - 1)(2ax + x^2),$$

the required equation.

158. To determine the points where the curve cuts the axis of x, make $y = 0$, then $x = 0$, or $x = -2a$; the value $x = 0$ gives the point A already known; the other value $-x = 2a = AA'$, determines the point A'. Bisect AA' in C, then in the equation to the hyperbola making $x = -a$, we get $y^2 = -(e^2 - 1)a^2$, or $y = \pm a\sqrt{e^2 - 1} \,.\, \sqrt{-1}$, which are imaginary values; hence the curve does not, as in the case of the ellipse, meet the line BB' drawn perpendicular to AA' through its middle point; if however we put $a\sqrt{e^2 - 1} = b$, and take BC, $B'C$, each equal to b, BB' will be denoted by $2b$; and the equation will become

$$y = \pm \frac{b}{a}\sqrt{2ax + x^2}.$$

159. In order to transfer the origin to C, we must change x into $x' - a$, since $AN = CN - CA$;

$$\therefore y^2 = \frac{b^2}{a^2}\{2a(x' - a) + (x' - a)^2\} = \frac{b^2}{a^2}(x'^2 - a^2).$$

This form of the equation shews that the origin is the centre of the hyperbola, and that the co-ordinate axes are Axes of the hyperbola (Art. 64); but the term, axis, is more particularly appropriated to the portions of those lines, $AA' = 2a$, $BB' = 2b$; the former of which meets the hyperbola and is called the *transverse* axis, and its extremities are called the vertices of the hyperbola; and the latter, although the line along which it is measured does not meet the curve, is taken for the second axis of the hyperbola, and is called the *conjugate* axis. Since $b = a\sqrt{e^2 - 1}$, where e may have any value greater than 1, b may be either greater or less than a.

160. To trace the hyperbola by means of its equation. The equation to the hyperbola referred to its axes is

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2};$$

hence for all values of x between $+a$ and $-a$, y is imaginary, and therefore no part of the curve lies in the space

bounded by two indefinite lines through A, A', parallel to BC. When $x = a$, $y = 0$, and as x increases positively from a to ∞, the two values of y are real and increase from zero to ∞, and give the infinite branch ZAz situated symmetrically with respect to Cx; and since when the sign of x is changed, the values of y do not alter, the negative values of x will give a branch $Z'A'z'$ precisely similar to the former, on the other side of BB', which is described by taking $SP' : P'M$ as $e : 1$. Moreover the two opposite branches of which the hyperbola is composed, will everywhere turn their convexities towards the axis BB'; otherwise a straight line might intersect them in more than two points, which is impossible, (as will appear Art. 186).

161. Since the hyperbola is symmetrically situated with respect to its axes, if we take $CH = CS$, $CX' = CX$, and draw kX' parallel to CB, the curve may be described by means of the focus H and directrix kX', exactly in the same way as by means of S and KX. Hence the hyperbola has two foci, situated in the transverse axis at equal distances from its centre.

Also since $a = \dfrac{p}{e-1}$, we have $AS = a(e-1)$;

$$\therefore\ SC = AC + AS = a + a(e-1) = ae, \text{ and } e = \frac{SC}{AC}.$$

The quantity e, which expresses the ratio of the distance between either focus and the centre, to the semi-transverse axis, is called the eccentricity. Since $b = a\sqrt{e^2-1}$, the eccentricity, e, expressed by the semi-axes, is equal to $\dfrac{\sqrt{a^2+b^2}}{a}$.

Hence $SC = \sqrt{a^2+b^2}$, and $AS \cdot A'S = BC^2$.

162. Since $AS = e \cdot AX$ (Art. 157), we have $AX = \dfrac{a(e-1)}{e}$;

$$\therefore\ CX = AC - AX = \frac{a}{e} = \frac{a^2}{ae} = \frac{CA^2}{CS},$$

$$\text{and } SX = AS + AX = \frac{a(e^2-1)}{e} = \frac{CB^2}{CS},$$

which determine the directrix relative to the centre, and focus.

The double ordinate passing through the focus is called the latus rectum. To find its value, make $x = CS = ae$ (fig. 53) in the equation to the hyperbola,

$$\therefore\ y^2 = \frac{b^2}{a^2}(a^2e^2 - a^2) = b^2(e^2-1) = \frac{b^4}{a^2};\ \text{(Art. 158)}$$

$$\therefore\ y = \pm\frac{b^2}{a},\quad \therefore\ LL' = \frac{2b^2}{a},\quad \text{or } = 2a(e^2-1).$$

163. If the distance CS between the focus and centre be supposed to become infinite, the distance AS between the focus and vertex remaining finite, the hyperbola will be changed into a parabola.

Since $\frac{b^2}{a^2} = e^2 - 1$, the equation reckoned from the vertex may be written

$$y^2 = 2a(e^2-1)x + (e^2-1)x^2,$$

$$\text{or } y^2 = 2p(e+1)x + (e^2-1)x^2, \text{ if } AS = p.$$

$$\text{But } e = \frac{SC}{AC} = \frac{SC}{SC-p} = \frac{1}{1-\frac{p}{SC}};\ \text{let } SC = \infty,\ \therefore\ e = 1;$$

$$\therefore\ y^2 = 4px, \text{ the equation to a parabola.}$$

164. Hence the equation $y^2 = 2p(1+e)x - (1-e^2)x^2$, is that to an ellipse, parabola, or hyperbola, according as $e <, =,$ or > 1; and therefore every conic section may be represented by the equation $y^2 = mx + nx^2$; and it will be a hyperbola, ellipse, or parabola, according as n is positive, negative, or zero. This is the simplest form of the equation by which the Conic Sections can be collectively represented.

When $b = a$, or $e^2 = 2$, the above equations to the hyperbola become (Arts. 158, 159)

$$y^2 = x^2 - a^2, \quad y^2 = 2ax + x^2.$$

The hyperbola in this case is called rectangular, and it is to the ordinary hyperbola what the circle is to the ellipse.

165. Since $y^2 = \frac{b^2}{a^2}(x^2 - a^2) = \frac{b^2}{a^2}(x + a)(x - a)$,

$$\text{gives } PN^2 = \frac{BC^2}{AC^2} . A'N . AN,$$

we see that the square of the ordinate varies as the rectangle of the distances of its foot from the extremities of the transverse axis.

166. The equation to the hyperbola results from that to the ellipse, by changing b^2 into $-b^2$, or b into $b\sqrt{-1}$. This remark may be of use in enabling us to foresee those properties of the hyperbola which are analogous to properties of the ellipse. In general, if any result in terms of its axes be obtained for the ellipse, the corresponding result for the hyperbola may be deduced by writing $b\sqrt{-1}$ for b.

167. To express the distances of any point in the hyperbola from the foci, in terms of its abscissa.

By Definition, (fig. 52)

$$SP = e . PM = e . XN = e . (CN - CX)$$

$$= e\left(x - \frac{a}{e}\right) = ex - a; \text{ (Art. 162)},$$

now e is greater than 1, and as long as P is in that branch of the hyperbola of which S is the interior focus, x is greater than a; therefore this expression for SP is always positive.

$$HP = e . Pk = e . (CN + CX') = e\left(x + \frac{a}{e}\right) = ex + a.$$

168. Exactly in the same way as for the ellipse, it may be shewn that the foci are the only points whose distances from every point in the curve, can be expressed rationally in terms of the abscissa of the point. (Art. 116).

169. Hence, subtracting,

$$HP - SP = 2a;$$

or the difference of the focal distances of any point in the hyperbola is constant, and equal to the transverse Axis.

Also the excess of the greater above the smaller focal distance of a point not in the hyperbola, will be $>$ or $< 2a$, according as it is situated on the concave or convex side of the curve.

In SP produced take a point Q on the same side of the conjugate axis as S is, and join HQ (fig. 56),

$$\text{then } HQ < HP + PQ, \quad SQ = SP + PQ,$$

$$\text{and } HQ \text{ is greater than } SQ;$$

$$\therefore HQ - SQ < HP - SP < 2a.$$

Again in SP take a point Q' and join HQ';

$$\text{then } HQ' + Q'P > HP, \quad SQ' + Q'P = SP;$$

$$\therefore HQ' - SQ' > HP - SP > 2a.$$

This property affords a simple method of determining any number of points in a hyperbola of which we know the transverse axis and foci. In $A'A$ (fig. 53) produced take any point F, and with centres S and H and radii respectively equal to AF, $A'F$, describe circles intersecting in P, P'; these are manifestly points in the hyperbola.

The curve may be described by a continuous motion if we have a rule HM moveable about the focus H, and a string SPM fastened to M and to the other focus S, of such a length that $HM - SPM = AA'$; then as HM revolves about H, if a point P slide along HM so as always to confine a

portion of string PM against it, the point will trace out a portion of the hyperbola; for we shall always have

$$HP - SP = HP + PM - (SP + PM) = HM - SPM = AA'.$$

This property also furnishes the following method of investigating the equation to the hyperbola.

170. To find the locus of a point the difference of whose distances from two fixed points is constant.

Through the two fixed points S, H, (fig. 53) draw the indefinite line Hx, bisect SH in C, and through C draw Cy perpendicular to it; and take Cx, Cy, for the axes of the co-ordinates, as the locus will evidently be symmetrical with respect to these lines. Let $SC = CH = c$, $CN = x$, $NP = y$, the co-ordinates of any point P, and $HP - SP = 2a$.

$$\text{Then } HP^2 = (CN + CH)^2 + NP^2 = (x + c)^2 + y^2,$$

$$SP^2 = (CN - CS)^2 + NP^2 = (x - c)^2 + y^2;$$

$$\therefore\ HP^2 - SP^2 \text{ or } 2a(HP + SP) = 4cx;$$

$$\therefore\ HP + SP = \frac{2cx}{a};$$

$$\text{but } HP - SP = 2a;$$

$$\therefore\ HP = \frac{cx}{a} + a;$$

$$\therefore\ \left(\frac{cx}{a} + a\right)^2 = (x + c)^2 + y^2,$$

$$\therefore\ \frac{c^2x^2}{a^2} + 2cx + a^2 = x^2 + 2cx + c^2 + y^2, \text{ or } y^2 = \frac{c^2 - a^2}{a^2}x^2 - c^2 + a^2,$$

$$\text{or } y^2 = \frac{c^2 - a^2}{a^2}(x^2 - a^2).$$

Now $HP - SP$ is less than SH, or $a < c$, consequently $c^2 - a^2$ is a positive quantity, and the above equation, as we should expect, represents a hyperbola whose transverse axis is $2a$, and conjugate axis $2\sqrt{c^2 - a^2}$.

171. To find the polar equation to the hyperbola, one of the foci being the pole.

Let the interior focus be the pole, and let the polar co-ordinates of any point P be $SP = r$, $\angle xSP = \theta$ (fig. 52), then $SN = r\cos\theta$, and $XN = XS + SN = \dfrac{a(e^2-1)}{e} + r\cos\theta$;

$$\therefore\ r = e\,.\,XN = a(e^2-1) + er\cos\theta,$$

$$\text{or } r(1 - e\cos\theta) = a(e^2-1),$$

$$\therefore\ r = \frac{a(e^2-1)}{1 - e\cos\theta}.$$

Since $e > 1$, there is some angle $xSD = \alpha$, (fig. 53) whose cosine $= \dfrac{1}{e}$; for values of θ less than α, r is negative, and there are no points in the branch AZ corresponding to those values, because $SP = ex - a$ is always positive. When $\theta = \alpha$, r is infinite and the radius vector meets the curve at an infinite distance; when θ exceeds α, r is positive, and as θ increases to π we get the portion of the curve ZA; when θ increases beyond π, the same values of r recur in an inverse order, giving the portion Az, till $\theta = 2\pi - \alpha$, when r is again infinite and afterwards becomes negative.

172. If S be the exterior focus, and the co-ordinates of any point P' be $SP' = r$, $\angle xSP' = \theta$, then $SN' = -r\cos\theta$,

$$\text{and } XN' = SN' - SX = -r\cos\theta - \frac{a}{e}(e^2-1),$$

$$\therefore\ r = e\,.\,XN' = -re\cos\theta - a(e^2-1),$$

$$\text{or } r = \frac{-a(e^2-1)}{1 + e\cos\theta}.$$

In this case also, r is negative and therefore has no point corresponding to it, till $\theta = \pi - \alpha$, when it becomes infinite, and then produces the branch of the hyperbola $Z'A'z'$ as θ changes from $\pi - \alpha$ to $\pi + \alpha$.

There is no difficulty in shewing that if we remove the restriction of having r positive, and measure negative values

upon the radius vector produced backwards, the same equation will represent both branches of the hyperbola.

173. To find the polar equation to the hyperbola, the centre being the pole.

Let $CP = r$, $\angle xCP = \theta$ (fig. 53), be the polar co-ordinates of a point P, whose rectangular co-ordinates are x and y; then $x = r\cos\theta$, $y = r\sin\theta$, and substituting in the equation $a^2y^2 - b^2x^2 = -a^2b^2$, we get

$$r^2(a^2\sin^2\theta - b^2\cos^2\theta) = -a^2b^2,$$

or, dividing by a^2 and observing that $\frac{b^2}{a^2} = e^2 - 1$,

$$r^2(1 - e^2\cos^2\theta) = -b^2;$$

$$\therefore\ r = \frac{b}{\sqrt{e^2\cos^2\theta - 1}}.$$

Taking θ from zero to α, where $\cos\alpha = \frac{1}{e}$, we get the part AZ; from $\theta = \alpha$ to $\theta = \pi - \alpha$, r is imaginary; from $\theta = \pi - \alpha$ to $\theta = \pi + \alpha$ we get the branch $Z'A'z'$; and the remaining portion Az, in taking θ from $2\pi - \alpha$ to 2π.

Tangent and Normal to the Hyperbola.

174. To find the equation to the tangent of a hyperbola at a given point.

The equation to the hyperbola being $a^2y^2 - b^2x^2 = -a^2b^2$, in order to find the equation to the tangent, the process will be the same as for the ellipse, with the sole difference that b^2 will every where be replaced by $-b^2$, and the result will be the same, subject to that modification. Hence if x', y', be the co-ordinates of the point of contact, and α the angle which the touching line makes with the axis of x,

$$\tan\alpha = \frac{b^2x'}{a^2y'},\ (\text{Art. } 121)$$

and the equation to the tangent will be $y - y' = \frac{b^2 x'}{a^2 y'}(x - x')$

$$\text{or } a^2 yy' - b^2 xx' = -a^2 b^2.$$

175. The formula $\tan\alpha = \frac{b^2}{a^2}\frac{x'}{y'}$,

since it does not alter when x' and y' are replaced by $-x'$ and $-y'$, shews that if PC (fig. 54) be produced to meet the hyperbola in P', the tangents at P and P' are parallel, as we might have foreseen on account of the symmetrical position of the hyperbola relative to its Axes; and that at the points A, A', for which $y' = 0$, $x' = \pm a$, $\tan\alpha = \infty$, or the tangents are perpendicular to the axis. Also, replacing x' by its value

$\frac{a}{b}\sqrt{y'^2 + b^2}$, derived from the equation $y'^2 = \frac{b^2}{a^2}(x'^2 - a^2)$

$$\text{we get } \tan\alpha = \frac{b}{a}\sqrt{1 + \frac{b^2}{y'^2}},$$

which shews that as y' increases from zero to ∞, $\tan\alpha$ diminishes from ∞ to a limit $\frac{b}{a}$.

176. To find where the tangent meets the transverse axis, make in the equation to the tangent, its ordinate $y = 0$;

$$\therefore x = \frac{a^2}{x'}, \text{ or } CT = \frac{CA^2}{CN}.$$

As this result is independent of b, it will be the same for all hyperbolas described with the same transverse axis.

Similarly, putting $x = 0$, to find where the tangent meets the conjugate axis, we get $y = -\frac{b^2}{y'}$, or $CT' . PN = BC^2$.

The value of CT diminishes as x' increases, but is always of the same sign, and becomes zero when $x' = \infty$. Hence when the point of contact is at an infinite distance, the tangent passes through the centre, and (Art. 175) makes with

the transverse axis an angle whose tangent is $\frac{b}{a}$; i. e. it coincides with the diagonal CW, of the rectangle constructed with the semi-axis, and has for its equation $y = \frac{b}{a}x$.

177. The subtangent $NT = CN - CT = x' - \frac{a^2}{x'} = \frac{x'^2 - a^2}{x'}$.

178. To find the equation to the normal of a hyperbola at a given point.

Let x', y', be the co-ordinates of the given point, then since the normal is perpendicular to the tangent at that point, it is easily seen that its equation is

$$y - y' = -\frac{a^2 y'}{b^2 x'}(x - x').$$

179. To determine the point G (fig. 54) where the normal meets the transverse axis, make in the equation to the normal its ordinate $y = 0$;

$$\therefore \; -y' = -\frac{a^2 y'}{b^2 x'}(x - x'),$$

$$\text{which gives } x = x'\left(1 + \frac{b^2}{a^2}\right), \text{ or } CG = e^2 . CN.$$

Hence, the least value of CG, when $x' = a$, is $\frac{a^2 + b^2}{a}$.

The subnormal $NG = CG - CN = \frac{b^2}{a^2} . x' = \frac{b^2}{a^2} . CN$.

180. The normal at any point bisects the exterior angle between the focal distances of that point.

For $SG = CG - CS = e^2 x' - ae = e . SP$, (Art. 167)

$HG = CG + CH = e^2 x' + ae = e . HP$;

$\therefore \frac{SG}{HG} = \frac{SP}{HP}$; $\therefore PG$ bisects the angle SPh. (Euc. VI. Prop. A.)

Also since GPT, GPt, are right angles, and $SPG = hPG$;

$$\therefore\ SPT = hPt = HPT,$$

or the focal distances make equal angles with the tangent on opposite sides of it. Hence if an ellipse and hyperbola have the same foci they will cut one another at right angles; for at the point of intersection the tangent to the hyperbola, since it bisects the angle between the focal distances, will coincide with the normal to the ellipse, and therefore be perpendicular to the tangent of the ellipse.

181. These properties furnish a simple method of drawing a tangent to the hyperbola through a given point.

First, let the point be in the curve as P (fig. 55).

Join SP, HP, make $HK = 2AC$, join SK and draw PY perpendicular to it; then in the triangles SPY, KPY, $PK = HP - 2AC = SP$, PY is common, and the angles at Y are right angles: $\therefore \angle SPY = KPY$, and consequently PY is a tangent at P.

182. Next let the point be on the convex side of the hyperbola (fig. 80 and 81). Join the proposed point T with the more remote focus H, and with centre H and radius $= 2AC$ describe a circle cutting HT or HT produced in O. Then if T falls within the circle, TS is less than TH, and S is necessarily outside the circle; but if T falls without the circle, $HT - ST < 2AC < HT - OT$, and therefore $ST > OT$; consequently in both cases a circle described with centre T and radius TS must intersect the former circle in two points K, K'. Join HK meeting the hyperbola in P and join TP, then TP is a tangent at P. For in the triangles TPK, TPS, $SP = HP \pm 2AC = PK$, $TS = TK$, and PT is common; $\therefore$ PT bisects the angle SPH and is therefore a tangent at P.

Similarly, if HK' be joined and produced to meet the hyperbola in Q, a second point of contact will be determined. The two tangents will belong to the same, or to opposite branches of the hyperbola, according as the point in which they intersect lies in $\angle LCl$ or its opposite, or in $\angle LCL'$ or its op-

posite (Art. 176); and the angles which the tangents subtend at either focus will in the former case be equal to one another, and in the latter supplementary to one another.

For in fig. 81, it is evident that $\angle TKH = TK'H$; $\therefore TKP = TK'Q$, $\therefore TSP = TSQ$, and $THP = THQ$. Again in fig. 80, $\angle TKH = TK'H$, $\therefore \pi - TKH = TK'Q$, or $\pi - TSP = TSQ$; and $\pi - THP = THK = THQ$.

183. The locus of the feet of the perpendiculars dropped from the foci upon the tangent to a hyperbola, is the circumference of the circle whose diameter is the transverse axis.

For joining CY (fig. 55), since SH is bisected in C, and SK in Y, CY is parallel to HK and $= \frac{1}{2}HK = AC$. Also, drawing HZ, CQ, perpendicular to ZY, Q is the middle point of ZY, and therefore $CZ = CY = CA$.

184. Since C is the centre of the circle which is the locus of Y and Z, and SYZ is a right angle, if SY and ZC be produced to meet in S', this will be a point in the circumference; and from the equal triangles SCS', HCZ, $SS' = HZ$;

$$\therefore SY \times HZ = SY \times SS' = SA \times SA' = BC^2 \text{ (Art. 161).}$$

Also since $\dfrac{SY}{SP} = \dfrac{HZ}{HP}$ (Art. 180), or $\dfrac{SY}{HZ} = \dfrac{SP}{HP}$, we have

$$\therefore SY^2 = BC^2 \times \frac{SP}{HP};$$

or, if SP, SY, be denoted respectively by r and p,

$$p^2 = b^2 \cdot \frac{r}{2a + r}.$$

185. Draw CE parallel to PY (fig. 55), then CP is a parallelogram, and $PE = CY = CA$.

All the properties of the ellipse proved in Arts. 132—136, may without difficulty be extended to the hyperbola.

The Hyperbola referred to its Conjugate Diameters.

186. To determine the intersection of a straight line with a hyperbola.

Exactly in the same way as for the ellipse, it may be shewn that the ordinates of the points of intersection of a straight line and hyperbola, whose equations are respectively

$$y = mx + c, \quad a^2y^2 - b^2x^2 = -a^2b^2,$$

are the roots of the equation

$$(b^2 - m^2a^2)\,y'^2 - 2b^2cy' + (c^2 - m^2a^2)\,b^2 = 0.$$

Hence the line cannot cut the curve in more than two points, and if the roots are impossible, it will not meet the curve at all. If the roots are equal, the line will touch the hyperbola; and we get $y = mx \pm \sqrt{m^2a^2 - b^2}$, for the equation to the tangent of the hyperbola, in terms of the angle which it makes with the transverse axis.

187. To find the locus of the middle points of a system of parallel chords.

Let the chords be parallel to a line CW through the centre (fig. 57), whose equation is $y = mx$; then the equation to any one of the chords QQ' will be $y = mx + c$, and to determine the points in which it meets the hyperbola, we must combine its equation with that to the hyperbola,

$$a^2y^2 - b^2x^2 = -a^2b^2;$$

this gives, eliminating x by the substitution $\frac{1}{m}(y - c)$,

$$y^2 - \frac{2b^2c}{b^2 - m^2a^2}\cdot y + b^2\cdot\frac{c^2 - m^2a^2}{b^2 - m^2a^2} = 0,$$

the roots of which will be represented by QM, $Q'M'$ if the values of y and the corresponding values of x, are all positive, but if one or both values of x are negative, the line will meet the opposite hyperbola; then if V be the middle point of QQ', and $CN = X$, $NV = Y$, its co-ordinates, $2NV = QM + Q'M'$;

$$\therefore\ Y = \frac{b^2 c}{b^2 - m^2 a^2},$$

$$\text{and } X = \frac{1}{m}(Y - c) = \frac{m a^2 c}{b^2 - m^2 a^2}.$$

Dividing one result by the other, in order to eliminate the quantity c which particularizes the chord, we get

$$Y = \frac{b^2}{m a^2} X,$$

a relation between the co-ordinates of the middle point of any chord, and therefore the equation to its locus, which is consequently a straight line CV passing through the origin.

The straight line which passes through the middle points of a system of parallel chords is called a diameter; hence all diameters of a hyperbola pass through its centre; and, conversely, every line through the centre may be considered as a diameter.

188. Hence, denoting the equation to any chord QQ' by $y = mx + c$, and the equation to the diameter CV which bisects it by $y = m'x$, we have

$$m' = \frac{b^2}{m a^2}, \text{ or } m m' = \frac{b^2}{a^2},$$

a simple relation, by means of which the equation of one may be deduced from that of the other.

189. If a diameter CV bisect the chords parallel to another diameter CW, then likewise the chords which are parallel to CV are bisected by CW.

For any one of the last-mentioned chords RR' may be represented by the equation $y = m'x + c'$; then the diameter which bisects it will have for its equation

$$y = \frac{b^2}{m' a^2} x, \text{ or } y = mx, \text{ which belongs to } CW.$$

Hence two diameters, whose equations $y = mx$, $y = m'x$, are so related that $mm' = \frac{b^2}{a^2}$, have the property that each bisects the chords parallel to the other; they are called Conjugate Diameters. But the term is usually restricted to those portions of them PP', DD', which are intercepted by the proposed hyperbola, and the hyperbola which is conjugate to it (fig. 58).

190. The latter is a hyperbola $BDB'D'$, whose transverse and conjugate axes are respectively equal to, and in the same straight line with, the conjugate and transverse axes of the proposed curve; and the employment of it is attended with great conveniences in stating and investigating the properties of the hyperbola.

The equation to the conjugate hyperbola, referred to the same axis of x and axis of y as the primitive hyperbola, so that $DM = y$, $CM = x$, (fig. 58) will consequently be

$$x^2 = \frac{a^2}{b^2}(y^2 - b^2), \quad \text{or } y^2 = \frac{b^2}{a^2}(x^2 + a^2),$$

which we observe results from the equation to the primitive hyperbola (Art. 158), by replacing a^2 and b^2 by $-a^2$ and $-b^2$.

191. If PT be a tangent at P (fig. 58), and x', y', the co-ordinates of P, then the equation to PT is

$$y - y' = \frac{b^2x'}{a^2y'}(x - x'); \text{ (Art. 174)}$$

but the equation to CP is $y = \frac{y'}{x'}x$, and therefore the equation to CD, the diameter conjugate to CP, is

$$y = \frac{b^2x'}{a^2y'}x,$$

which represents a line parallel to PT. Hence the tangent at the extremity of any diameter, is parallel to the corre-

sponding conjugate diameter. Similarly, the tangent to the conjugate hyperbola at D is parallel to CP; and if tangents be applied to the hyperbola and its conjugate, at the extremities of a pair of conjugate diameters, they will form a parallelogram inscribed in the two curves, whose sides will be bisected in the points of contact.

192. Of any two conjugate diameters, only one can meet the hyperbola.

Let $y = mx$ be the equation to a diameter; to determine its intersection with the curve, put mx for y in the equation

$$a^2y^2 - b^2x^2 = -a^2b^2,$$

and we find for the abscissæ of the points of intersection

$$x = \pm\sqrt{\frac{a^2b^2}{b^2 - m^2a^2}},$$

which values are real as long as m is less than $\frac{b}{a}$, but imaginary if m be greater than $\frac{b}{a}$; in the former case the diameter intersects the curve, in the latter it does not. But the relation $mm' = \frac{b^2}{a^2}$ shews that if m be less than $\frac{b}{a}$, m' is greater than $\frac{b}{a}$; hence every diameter which meets the hyperbola, has its conjugate diameter amongst those which do not meet it.

193. If we construct on the axes of the curve, the rectangle Ll' (fig. 59), all the diameters which fall within the angle LCl, make with AC an angle whose tangent (abstracting the sign) is less than $\frac{b}{a}$; whilst the diameters which fall within the angle LCL' make with AC an angle whose tangent exceeds $\frac{b}{a}$; the former are those that meet the curve, the latter those that do not.

In the particular case when $m = \frac{b}{a}$, we have also $m' = \frac{b}{a}$, and the conjugate diameters coincide with Ll'; and as the value of x (Art. 192) becomes infinite, they meet the curve only at an infinite distance; similarly, when $m = -\frac{b}{a}$, we have $m' = -\frac{b}{a}$, and the two diameters coincide with the other diagonal $L'l$, and meet the curve only at an infinite distance. The lines Ll', $L'l$, are, for this reason, called Asymptotes; and they correspond to the equal conjugate diameters in the ellipse.

194. Having given the co-ordinates of the extremity of any diameter, to find those of the diameter conjugate to it.

Let CD be conjugate to CP (fig. 58), and let it meet the conjugate hyperbola in D; let x', y', be the co-ordinates of P, and consequently $y = \frac{y'}{x'}x$ the equation to CP, then $y = \frac{b^2x'}{a^2y'}x$ is the equation to CD; and to determine the co-ordinates of D we must combine this equation with the equation to the conjugate hyperbola, which is

$$a^2y^2 - b^2x^2 = a^2b^2.$$

This gives, eliminating y by the substitution $\frac{b^2x'}{a^2y'}x$,

$$\frac{b^4x'^2}{a^2y'^2}x^2 - b^2x^2 = a^2b^2,$$

$$\text{or } x^2(b^2x'^2 - a^2y'^2) = a^4y'^2, \text{ or } x^2 = \frac{a^2y'^2}{b^2};$$

$$\therefore\ x = CM = \frac{ay'}{b}, \text{ and } y = DM = \frac{bx'}{a};$$

the other pair of values of x and y belonging to the point D'.

195. The difference of the squares of any two semi-conjugate diameters, is equal to the difference of the squares of the semi-axes.

$$CP^2 = x'^2 + y'^2,$$

$$CD^2 = \frac{a^2y'^2}{b^2} + \frac{b^2x'^2}{a^2} = x'^2 - a^2 + y'^2 + b^2, \text{ because } y'^2 = \frac{b^2}{a^2}(x'^2 - a^2);$$

$$\therefore CP^2 - CD^2 = a^2 - b^2.$$

196. All parallelograms whose sides touch a hyperbola and the conjugate hyperbola at the extremities of a pair of conjugate diameters, are equal to one another.

Draw PF perpendicular to CD (fig. 58), then (Art. 191)

area of whole parallelogram $= 4CD \,.\, PF = 4CD \,.\, CT \sin TCF$

$$= 4DM \,.\, CT = 4\frac{bx'}{a} \,.\, \frac{a^2}{x'} = 4ab.$$

197. Draw the diagonal CL (fig. 58), which will pass through the middle point of DP, whose co-ordinates are equal to $\frac{1}{2}(CM + CN)$, $\frac{1}{2}(DM + PN)$;

$$\therefore \tan LCA = \frac{\frac{1}{2}\left(y' + \frac{bx'}{a}\right)}{\frac{1}{2}\left(x' + \frac{ay'}{b}\right)} = \frac{b}{a};$$

which value is independent of the positions of CP and CD; hence the parallelograms whose sides touch a hyperbola and its conjugate at the extremities of a pair of conjugate diameters, are not only equal in area, but they all have their diagonals in the same line; namely, the diagonal of the rectangle whose sides are the semi-axes.

198. If we denote CP, CD, by a', b', and $\angle PCD$ by γ, we have

$PF = a' \sin \gamma$, and $\therefore a'b' \sin \gamma = CD \times PF = ab.$

Also if we denote PF by p, we have the relation between the central distance of any point and the perpendicular from the centre upon the tangent at that point,

$$p^2 = \frac{a^2 b^2}{CD^2} = \frac{a^2 b^2}{a'^2 - a^2 + b^2}.$$

The magnitude and position of two conjugate diameters that include a given angle, may be determined in the same manner as for the ellipse (Art. 145).

199. The rectangle contained by the focal distances of any point, is equal to the square of the corresponding semi-conjugate diameter.

$$\begin{aligned} CD^2 &= CP^2 - a^2 + b^2 \\ &= x^2 + \frac{b^2}{a^2} x^2 - b^2 - a^2 + b^2 \\ &= e^2 x^2 - a^2 = (ex + a) \, . \, (ex - a) \\ &= SP \, . \, HP. \end{aligned}$$

200. To find the equation to the hyperbola referred to the system of oblique axes formed by any pair of conjugate diameters.

The equation to a hyperbola referred to its centre and axes is

$$a^2 y^2 - b^2 x^2 = - a^2 b^2.$$

Let the conjugate diameters CP, CD (fig. 60), be the new axes of x' and y', inclined to the axis of x at angles $PCA = \alpha$, $DCA = \beta$; then since the origin remains unaltered, the formulæ for passing from the rectangular to the oblique axes are (Art. 42)

$$x = x' \cos \alpha + y' \cos \beta, \qquad y = x' \sin \alpha + y' \sin \beta.$$

Hence, substituting and reducing,

$$(a^2 \sin^2 \alpha - b^2 \cos^2 \alpha) x'^2 + (a^2 \sin^2 \beta - b^2 \cos^2 \beta) y'^2 \\ + 2x'y' (a^2 \sin \alpha \sin \beta - b^2 \cos \alpha \cos \beta) = - a^2 b^2;$$

but $\tan\alpha.\tan\beta=\frac{b^2}{a^2}$; therefore $a^2\sin\alpha.\sin\beta-b^2\cos\alpha.\cos\beta=0$.

Also if $CP=a'$, $CD=b'$, we have (Arts. 178 and 190)

$$a'^2(a^2\sin^2\alpha-b^2\cos^2\alpha)=-a^2b^2,$$

$$b'^2(a^2\sin^2\beta-b^2\cos^2\beta)=+a^2b^2;$$

hence, substituting and dividing by $-a^2b^2$, we get for the required equation

$$\frac{x'^2}{a'^2}-\frac{y'^2}{b'^2}=1;$$

or, in a geometrical form, supposing PC produced to meet the curve in G,

$$QV^2=\frac{CD^2}{CP^2}PV.VG.$$

201. This equation, which, suppressing the accents of the variables, is

$$a'^2y^2-b'^2x^2=-a'^2b'^2,$$

being of precisely the same form as that relative to the axes, it follows that all properties which do not depend upon the inclination of the co-ordinates, will be common to the axes of the hyperbola and to its conjugate diameters.

Hence the equation to the tangent at a point $Q(x', y')$ will be

$$a'^2yy'-b'^2xx'=-a'^2b'^2,$$

and if the tangent meet the axis of the abscissæ in T, we shall have $CT=\frac{CP^2}{CV}$, as before; and if we wish to draw a tangent through an external point $Q(h, k)$ (fig. 61), we shall have, to determine the points of contact (x', y'), the equations

$$a'^2y'^2-b'^2x'^2=-a'^2b'^2,$$

$$a'^2ky'-b'^2hx'=-a'^2b'^2;$$

the latter, considering x' and y' as the variables, being the equation to the chord joining the two points of contact; and if we construct this line by taking

$$CT = \frac{a'^2}{h}, \quad CR = \frac{-b'^2}{k},$$

and joining RT, it will cut the hyperbola in the two points of contact.

202. Since the distance CT is independent of k, if through Q we draw a line parallel to CD, and from any other point in this line we draw a pair of tangents to the hyperbola, the secant passing through the new points of contact will cut the diameter CP in T, as this point only changes when h changes. Hence, if from the several points of any straight line, pairs of tangents be drawn to a hyperbola, the straight lines which join the corresponding points of contact will all intersect in the same point; and conversely if through any point we draw different chords and apply two tangents at the extremities of each, the locus of the intersection of the tangents will be a straight line.

203. The tangents at the extremities of any chord will intersect in the diameter of which the chord is an ordinate.

For, taking that diameter and its conjugate as the axes of x and y, the equation to the tangent will be

$$\pm a'^2 yy' - b'^2 xx' = -a'^2 b'^2,$$

according as we consider the point $Q(x', y')$, or the other extremity of the chord Q' whose co-ordinates are x', $-y'$, to be the point of contact; and in both cases when $y = 0$, $x = \frac{a'^2}{x'}$; therefore the tangents meet the axis of x in the same point T. (fig. 60).

Exactly in the same manner as for the ellipse (Art. 152), it may be shewn that if from the extremities of any diameter two chords be drawn to any point in a hyperbola, and one of them be parallel to a diameter, the other will be parallel to the conjugate diameter.

204. If from any point within or without a hyperbola, two lines be drawn parallel to two given straight lines to meet the curve, the rectangles of the segments will be to each other in an invariable ratio.

Let O (fig. 62) be the given point with co-ordinates h and k; then taking O for the pole, and measuring θ from a line parallel to the transverse axis, the polar equation to the hyperbola will be

$$a^2 (r \sin\theta + k)^2 - b^2 (r \cos\theta + h)^2 = -a^2 b^2,$$

which is of the form $r^2 + Mr - N = 0$,

$$\text{where } N = \frac{-a^2 b^2 - a^2 k^2 + b^2 h^2}{a^2 \sin^2\theta - b^2 \cos^2\theta} = r' r'',$$

if these be the two values of r.

Now let Pp, Qq be drawn parallel to CP', CQ' which make angles α, β, with Cx, then

$$PO \times Op : QO \times Oq :: a^2 \sin^2\beta - b^2 \cos^2\beta : a^2 \sin^2\alpha - b^2 \cos^2\alpha$$
$$:: CP'^2 : CQ'^2, \text{ (Art. 173)}$$

which ratio is independent of the position of the point O.

Hence if we suppose Pp, Qq, to move parallel to themselves till they become tangents to the hyperbola at points P and Q respectively, and intersect in a point O outside the curve, we have

$$OP : OQ :: CP' : CQ'.$$

The Hyperbola referred to its Asymptotes.

205. The diameters which never meet the hyperbola at any finite distance, are called Asymptotes.

These diameters coincide with the diagonals of the rectangle constructed with the semi-axes (Art. 193); and we shall now shew, according to the strict notion of an asymptote, that although they never meet the curve they approach indefinitely near to it. For, the equations to CL (fig. 59),

and to the hyperbola, when referred to the axes of the curve, being, respectively, (Arts. 19 and 159)

$$y = \frac{b}{a}x, \quad y = \frac{b}{a}\sqrt{x^2 - a^2},$$

the difference of the ordinates

$$PP' = \frac{b}{a}(x - \sqrt{x^2 - a^2}) = \frac{ab}{x + \sqrt{x^2 - a^2}};$$

therefore, as x increases, this difference continually diminishes, and ultimately vanishes when $x = \infty$. Similarly, it may be shewn that CL', whose equation is $y = -\frac{b}{a}x$, approaches indefinitely near to the other branch of the hyperbola.

It appears, by Art. 176, that the asymptote to the hyperbola, is the limiting position of the tangent, when the point of contact is infinitely distant.

206. When the hyperbola is referred to a pair of conjugate diameters, the directions of its asymptotes will be determined by the diagonals of the parallelogram constructed with the diameters; for those diagonals always coincide with the diagonals of the rectangle constructed with the semi-axes (Art. 197). Also in this case, where the co-ordinates are oblique, we may shew, exactly in the same manner as for rectangular co-ordinates, that the diagonals approach indefinitely near to the curve. For, the equation to the diagonal CL (fig. 63), and to the hyperbola, referred to the conjugate diameters CP, CD, are, respectively,

$$y = \frac{b'}{a'}x, \quad y = \frac{b'}{a'}\sqrt{x^2 - a'^2};$$

$$\therefore\ RQ = \frac{b'}{a'}(x - \sqrt{x^2 - a'^2}) = \frac{a'b'}{x + \sqrt{x^2 - a'^2}}.$$

Hence when $x = \infty$, RQ becomes zero; hence Ll' approaches indefinitely near to the portions PQ, $P'Q'$; and similarly it may be shewn that $L'l$ approaches indefinitely near to the other portions of the curve.

207. If any chord of a hyperbola be produced to meet the asymptotes, the parts of it intercepted between the curve and the asymptotes will be equal.

Let Qq (fig. 63), any chord, when produced cut the asymptotes in R, r; bisect Qq in V, join CV cutting the hyperbola in P, and refer the hyperbola to the diameter CP and its conjugate CD; then the equations to CR, Cr, are

$$y = \frac{b'}{a'}x, \quad y = -\frac{b'}{a'}x;$$

$\therefore$ $VR = Vr$, and $VQ = Vq$; $\therefore$ subtracting $QR = qr$.

Also $RQ \cdot Qr = (RV + VQ)(RV - VQ) = RV^2 - QV^2$

$$= \frac{b'^2}{a'^2}\{x^2 - (x^2 - a'^2)\} = b'^2 = CD^2;$$

i. e. the rectangle of the segments into which a line, terminated by the asymptotes, is divided by the curve, is equal to the square of the semi-diameter to which it is parallel.

If a line be drawn through P parallel to Rr, it will be a tangent at P, and $PL = Pl$. Hence every tangent terminated by the asymptotes is bisected in the point of contact.

208. From any point P (fig. 63) of the hyperbola, draw parallels PG, PF to the asymptotes, and draw the tangent Ll which is bisected in P. Then the parallelogram GF is half of the triangle LCl.

But area LCl is constant, whatever be the position of P, and equals ab (Art. 196).

Hence, denoting by 2α the angle between the asymptotes, and by x, y, the co-ordinates of P referred to the asymptotes as axes, so that $CF = x$, $FP = y$, we get

$$xy \sin 2\alpha = \frac{ab}{2}; \text{ but } \sin 2\alpha = \frac{2 \tan \alpha}{1 + \tan^2 \alpha} = \frac{2ab}{a^2 + b^2},$$

$$\text{since } \tan\alpha = \frac{b}{a};$$

$$\therefore\ xy = \tfrac{1}{4}(a^2 + b^2),$$

the equation to the hyperbola referred to its asymptotes, which may be likewise obtained in the following manner.

209. To transform the equation to the hyperbola referred to its axes, into that representing the hyperbola referred to its asymptotes.

Let the inferior asymptote be the axis of x'; and let $CM = x'$, $MP = y'$, (fig. 59) be the co-ordinates of a point referred to the asymptotes, and $CN = x$, $NP = y$, the co-ordinates of the same point referred to the axes of the hyperbola; and $\angle LCA = lCA = \alpha$; then drawing MQ, MR, respectively perpendicular to CN, and to PN produced, we have

$$x = NQ + QC = y'\cos\alpha + x'\cos\alpha,$$

$$y = PR - RN = y'\sin\alpha - x'\sin\alpha;$$

hence, substituting in the equation $a^2y^2 - b^2x^2 = -a^2b^2$, we get

$$a^2\sin^2\alpha\,(y' - x')^2 - b^2\cos^2\alpha\,(y' + x')^2 = -a^2b^2;$$

but $\tan\alpha = \dfrac{b}{a}$; and $\therefore\ a^2\sin^2\alpha = b^2\cos^2\alpha = \dfrac{b^2}{1 + \dfrac{b^2}{a^2}} = \dfrac{a^2b^2}{a^2 + b^2}$;

$$\therefore\ \frac{a^2b^2}{a^2 + b^2}\,.\,4x'y' = a^2b^2, \quad \text{or } x'y' = \frac{a^2 + b^2}{4}.$$

If $a = b$, $\tan\alpha = 1$, $2\alpha = \frac{1}{2}\pi$; and the asymptotes are then perpendicular to one another. The hyperbola with equal axes is therefore called rectangular, and its equation is $xy = \frac{1}{2}a^2$.

210. To find the equation to the line touching the hyperbola at a given point, when referred to its asymptotes.

Let the co-ordinates of the given point be x', y', and those of a point near it x'', y''; the equation to the line passing through them will be

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x'),$$

$$\text{but } x'y' = \tfrac{1}{4}(a^2 + b^2), \quad x''y'' = \tfrac{1}{4}(a^2 + b^2);$$

$$\therefore\ x''y'' - x'y' = 0, \quad \text{or } x'(y'' - y') + y''(x'' - x') = 0;$$

$$\therefore\ \frac{y'' - y'}{x'' - x'} = -\frac{y''}{x'}.$$

Hence the equation to the secant becomes

$$y - y' = -\frac{y''}{x'}(x - x');$$

and in order that it may become a tangent, we must suppose $y'' = y'$, which gives, for the equation to the tangent,

$$y - y' = -\frac{y'}{x'}(x - x'), \quad \text{or } x'y + y'x = 2x'y' = \tfrac{1}{2}(a^2 + b^2).$$

To find where the tangent cuts the asymptotes, make $y = 0$; $\therefore\ x = 2x'$, or $Cl = 2CF$, and $Ll = 2LP$ (fig. 63), agreeably to Art. 207.

211. To find the area $PNMQ$ contained between a hyperbola, its asymptote, and two ordinates to the asymptote.

Let the equation to the hyperbola be $xy = a^2$, and $\angle yAx = \omega$ (fig. 76), $AN = a$, $AM = b$; take x such that $\left(\frac{x}{a}\right)^n = \frac{b}{a}$, or $\frac{x}{a} = \sqrt[n]{\frac{b}{a}}$, and take the abscissæ in geometrical progression, so that

$$AN_1 = x, \quad AN_2 = \frac{x^2}{a}, \quad AN_3 = \frac{x^3}{a^2}, \text{ \&c.,} \quad AM = \frac{x^n}{a^{n-1}} = b;$$

and complete the (n) parallelograms PN_1, P_1N_2, &c.; then as n increases, $\frac{x}{a}$ tends continually to 1, and therefore the difference between any two consecutive abscissæ continually diminishes; and consequently the limit of the sum of the parallelograms, when n is infinite, is the hyperbolic area $PNMQ$.

But area of parallelogram $PN_1 = (x - a)\, a \sin\omega$,

area of $P_1N_2 = \left(\frac{x^2}{a} - x\right)\frac{a^2}{x}\sin\omega = (x - a)\, a\sin\omega$,

area of $P_2N_3 = \left(\frac{x^3}{a^2} - \frac{x^2}{a}\right)\frac{a^3}{x^2}\sin\omega = (x - a)\, a\sin\omega$,

....

therefore the sum of the parallelograms $= n\,(x - a)\, a\sin\omega$

$$= a^2\sin\omega . n\left\{\left(\frac{b}{a}\right)^{\frac{1}{n}} - 1\right\};$$

$\therefore$ hyperbolic area $PNMQ = a^2\sin\omega\,.\,\text{limit}\, n\left\{\left(\frac{b}{a}\right)^{\frac{1}{n}} - 1\right\} (n = \infty)$

$$= a^2\sin\omega\,.\,\text{limit}\; n\left\{(1 + \frac{b}{a} - 1)^{\frac{1}{n}} - 1\right\}$$

$$= a^2\sin\omega\left\{\left(\frac{b}{a} - 1\right) + \tfrac{1}{2}\left(\frac{1}{n} - 1\right)\left(\frac{b}{a} - 1\right)^2\right.$$

$$\left. + \frac{1}{2\,.\,3}\left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)\left(\frac{b}{a} - 1\right)^3 + \&\text{c.}\right\} (n = \infty)$$

$$= a^2\sin\omega\left\{\frac{b}{a} - 1 - \tfrac{1}{2}\left(\frac{b}{a} - 1\right)^2 + \tfrac{1}{3}\left(\frac{b}{a} - 1\right)^3 - \&\text{c.}\right\}$$

$$= a^2\sin\omega\log\left(\frac{b}{a}\right).$$

If $\omega = \frac{1}{2}\pi$, and $a = 1$, then area $PNMQ = \log AM$; or the area is the logarithm of its abscissa; on this account, the Napierian logarithms are sometimes called Hyperbolic.

SECTION IX.

ON THE SECTIONS OF THE CONE AND CYLINDER.

212. THE surface described by an indefinite straight line which is carried round the perimeter of a given circle, always passing through a fixed point, is called a cone (fig. 64).

The circle is called the base of the cone, and the fixed point its vertex, and the line joining the vertex and centre of the base is called the axis. The cone is moreover right or oblique, according as the axis is at right angles, or inclined, to the plane of the base.

As the generating line is unlimited in both directions from the vertex, the surface of the cone is composed of two portions or sheets, perfectly similar, situated on opposite sides of the vertex. Also from the mode of generation it follows that every plane parallel to the base will cut the cone in a circle; and every plane through the axis will cut it in two straight lines. When the surface is a right cone, every generating line will make the same angle with the axis.

The different curves obtained by cutting a cone by a plane are called Conic Sections.

Sections of a Right Cone by a Plane.

213. All sections of a right cone made by a plane are curves of the second order.

Let PAP' (fig. 65) be a section of a right cone made by any plane; and through the axis VO draw a plane perpendicular to that of the section, cutting the cone in the lines VB, VD, and the plane of the section in the line AN, which take for the axis of x. Through any point P of the curve AP draw a plane perpendicular to the axis, intersecting the cone in the circle MPQ, and the plane of

the section in PP'; then MQ will be a diameter of the circle, and PN will be at right angles to both AN and NM, and will consequently be a common ordinate to the circle and conic section AP. Draw Ay parallel to PN and take it for the axis of y, and let $AN = x$, $PN = y$ be rectangular co-ordinates of P; and choosing the data so as to embrace every case, and therefore not assuming that AN meets VQ produced, let $AV = d$, $\angle VAN = \theta$, $AVQ = 2\alpha$. Then since PN is perpendicular to the diameter MQ,

$$y^2 = MN \times NQ.$$

$$\text{But } \frac{MN}{AN} = \frac{\sin\theta}{\cos\alpha}; \quad \therefore MN = \frac{x\sin\theta}{\cos\alpha}.$$

And drawing NF parallel to QV,

$$\text{since } \angle ANF = GFN - GAN = VAG - (VAN - VAG)$$

$$= 2\,VAG - VAN = \pi - 2\alpha - \theta,$$

$$\frac{AF}{AN} = \frac{\sin(2\alpha+\theta)}{\cos\alpha}; \quad \therefore AF = \frac{x\sin(2\alpha+\theta)}{\cos\alpha};$$

$$\therefore NQ = 2d.\sin\alpha - \frac{x.\sin(2\alpha+\theta)}{\cos\alpha};$$

$$\therefore y^2 = \frac{2d.\sin\alpha.\sin\theta}{\cos\alpha}x - \frac{\sin\theta.\sin(2\alpha+\theta)}{\cos^2\alpha}x^2,$$

the equation to a curve of the second order; therefore every conic section is a curve of the second order; and it will be an ellipse, hyperbola, or parabola, according as the second term is negative, or positive, or zero, (Art. 164).

Now the second term can only change its sign when $\sin(2\alpha+\theta)$ changes its sign. Hence the section will be an ellipse as long as $2\alpha+\theta$ is less than π, and therefore AN meets VQ produced, or the cutting plane meets only one sheet of the cone.

It will be a hyperbola when $2\alpha+\theta$ is greater than π, and therefore AN and VQ intersect when produced backwards, and the cutting plane meets both the sheets of the cone.

It will be a parabola when $2\alpha + \theta = \pi$, and therefore AN, VQ are parallel, or the cutting plane is parallel to a generating line of the cone.

214. To determine the axes of the conic section, we have, since the co-ordinates are rectangular, by comparing the equation (supposing it to represent an ellipse and therefore $\sin(2\alpha + \theta)$ to be positive)

$$y^2 = \frac{2d \sin\alpha . \sin\theta}{\cos\alpha} x - \frac{\sin\theta . \sin(2\alpha + \theta)}{\cos^2\alpha} x^2$$

with $y^2 = \dfrac{2b^2}{a} x - \dfrac{b^2}{a^2} x^2$,

the latus rectum or $\dfrac{2b^2}{a} = 2d . \tan\alpha . \sin\theta$,

$$\text{and } \frac{b^2}{a^2} = \frac{\sin\theta . \sin(2\alpha + \theta)}{\cos^2\alpha};$$

$$\therefore\ 2a = \frac{2d \sin\alpha . \cos\alpha}{\sin(2\alpha + \theta)};$$

$$\therefore\ 2b^2 = \frac{2d^2 \sin^2\alpha \sin\theta}{\sin(2\alpha + \theta)},\ \text{or } b = d . \sin\alpha \sqrt{\frac{\sin\theta}{\sin(2\alpha + \theta)}}.$$

215. The minor axis may however be more conveniently expressed in the following manner.

From the extremities of the axis major let fall perpendiculars $AF = f$, $A'G = g$ (fig. 64), upon the axis of the cone; and through C, the middle point of AA', draw a plane parallel to the base, cutting the section in BB' which is its minor axis, and the cone in the circle MBQ; then

$$BC^2 = MC \times CQ = A'G \times AF = fg,$$

because MC, being parallel to DA', $= \frac{1}{2} DA' = A'G$

and similarly $CQ = AF$.

Hence the distance of the foci of the elliptic section $= AD$; for, dropping the perpendicular AE, $A'E = f + g$;

$$\therefore\ AD^2 = 4a^2 + 4g^2 - 4g(f+g) = 4a^2 - 4fg = 4(a^2 - b^2);$$

$$\therefore\ AD = 2\sqrt{a^2 - b^2} = \text{distance of foci.}$$

216. If in that section of a cone through the axis which is perpendicular to the plane of an elliptic section, we describe circles touching the generating lines of the cone and the axis of the section, the points of contact with the axis will be the foci of the section.

For the distance of the foci $= A'D'$ (fig. 66).

$$\text{But } A'D' = A'U' - D'U' = A'S - AU$$
$$= AA' - 2AS;$$
$$\therefore\ AS = \tfrac{1}{2}(AA' - A'D');$$

therefore S is a focus. Similarly, H may be shewn to be the other focus.

Produce UU' to meet AA' produced in X,

then from the similar triangles AUX, ADA',

$$\frac{AX}{AA'} = \frac{AU}{AD} \text{ or } \frac{AX}{2AC} = \frac{AS}{2SC};$$

$$\therefore\ \frac{AX}{AC} = \frac{AS}{SC} \text{ or } \frac{CX}{AC} = \frac{AC}{SC};$$

therefore X is the point where the directrix meets the axis (Art. 110). Similarly, X' is the point where the other directrix meets the axis.

217. When the section is a hyperbola, the equation is

$$y^2 = \frac{2d \sin\theta \, . \sin\alpha}{\cos\alpha} x - \frac{\sin\theta \, . \sin(2\alpha + \theta)}{\cos^2\alpha} x^2,$$

where $\sin(2\alpha + \theta)$ is a negative quantity, and consequently the second term is positive; by comparing this with

$$y^2 = \frac{2b^2}{a}x + \frac{b^2}{a^2}x^2$$

we may determine the axes of the curve, as in the case of the ellipse. When in this case $d = 0$, the equation becomes

$$y^2 = -\frac{\sin\theta . \sin(2\alpha + \theta)}{\cos^2\alpha}x^2,$$

which represents two generating lines of the cone.

In this case also, the semi-conjugate axis is a mean proportional between the perpendiculars dropped from the vertices of the hyperbola upon the axis of the cone; and the distance of its foci is equal to the portion of the slant side intercepted by the perpendiculars.

$$\text{For } b^2 = \frac{d^2\sin^2\alpha\sin\theta}{-\sin(2\alpha+\theta)} = d\sin^2\alpha . \frac{d\sin\theta}{\sin(2\pi - 2\alpha - \theta)}$$

$$= AV . A'V . \sin^2\alpha = AF . A'G = fg \text{ (fig. 82)};$$

$$\text{and } AD^2 = 4a^2 + 4g^2 - 4g(g-f) = 4a^2 + 4fg = 4(a^2 + b^2).$$

218. When the section is a parabola, or $2\alpha + \theta = \pi$, the equation is, since $\sin\theta = \sin 2\alpha$,

$$y^2 = \frac{2d\sin\theta . \sin\alpha}{\cos\alpha}x = 4d\sin^2\alpha x.$$

219. We must now demonstrate the converse proposition, namely, that curves of the second order are conic sections.

Every curve of the second order is contained in the equation

$$y^2 = 4px + nx^2,$$

where $4p$ is the latus rectum, and n the square of the ratio of the axes, abstracting the sign. What we have to demonstrate is, that the quantities p, n, and α being given, we

can assign real values of d and θ which shall render the above equation identical with

$$y^2 = \frac{2d \sin\alpha . \sin\theta}{\cos\alpha} x - \frac{\sin\theta . \sin(2\alpha + \theta)}{\cos^2\alpha} x^2.$$

Equating the coefficients of x and x^2 in the two equations, we get

$$\frac{d \sin\alpha . \sin\theta}{\cos\alpha} = 2p, \quad \frac{\sin\theta . \sin(2\alpha + \theta)}{\cos^2\alpha} = -n,$$

the former of which will give a real value of d when θ is real; the latter may be transformed into

$$\tfrac{1}{2}\{\cos 2\alpha - \cos(2\alpha + 2\theta)\} = -n\cos^2\alpha,$$

$$\text{or } \cos 2(\alpha + \theta) = 2(1 + n)\cos^2\alpha - 1.$$

In the ellipse, n is negative and less than 1; hence the preceding value of $\cos 2(\alpha + \theta)$ lies between $+1$ and -1, and therefore θ is always real; consequently any given ellipse may be regarded as a section of any proposed right cone whatever.

In the hyperbola, n is positive and of any magnitude; if the above value of $\cos 2(\alpha + \theta)$ be negative, it will be evidently less than 1, and θ will be real; but if it be positive, we must have, in order that θ may be real,

$$2(1 + n)\cos^2\alpha - 1 \text{ less than } 1,$$

$$\text{and } \therefore \cos\alpha \text{ less than } \frac{1}{\sqrt{1+n}}, \text{ or than } \frac{a}{\sqrt{a^2 + b^2}};$$

but if ω be the angle which the asymptote makes with the transverse axis, $\cos\omega = \dfrac{a}{\sqrt{a^2 + b^2}}$; $\therefore \cos\alpha < \cos\omega$, $\therefore \alpha > \omega$; and therefore, in order that a given hyperbola may be cut from a given cone, the vertical angle of the cone must be not less than the angle between the asymptotes.

In the parabola, $n=0$; therefore $\sin\theta = 0$, or $\sin(2\alpha+\theta) = 0$; the first is inadmissible, for it makes $p = 0$; the second gives

$2\alpha + \theta = \pi$, which will always furnish a real value for θ; hence a given parabola may be cut from any proposed cone.

Sections of Cylinder and Oblique Cone by a Plane.

220. To determine the curve which results from the intersection of a right cylinder with a plane.

Let APA' (fig. 67) be a section of a right cylinder, $AA'D$ a section of the cylinder through its axis, perpendicular to the plane of the section. Through any point P draw a plane perpendicular to the axis of the cylinder, intersecting it in a circle whose diameter is MQ, and the plane of the section in PP' which will be perpendicular to MQ, AA', and will be a common ordinate of the section and circle.

Let $AN = x$, $NP = y$, $AA' = 2a$, $AD = 2r$,

$$\text{then } y^2 = MN \,.\, NQ;$$

$$\text{but } \frac{MN}{AD} = \frac{AN}{AA'}, \text{ or } MN = \frac{r}{a}x,$$

$$\frac{NQ}{AD} = \frac{NA'}{AA'}, \text{ or } NQ = \frac{r}{a}(2a - x);$$

$$\therefore\ y^2 = \frac{r^2}{a^2}(2ax - x^2),$$

the equation to an ellipse.

221. In the same manner the nature of the sections of an oblique cone may be determined; but this, as well as the discussion of the sections of Conoids or figures generated by the revolution of conic sections about their axes, may be more conveniently deferred to Geometry of Three Dimensions. There is however one important property of the oblique cone which admits of a simple demonstration, viz. that it may be cut by other planes besides those parallel

to its base, so that the sections may be circles, and which we shall give here.

Let VBD (fig. 68) be the principal section of an oblique cone, that is, a section made by a plane through its axis perpendicular to its base; and let MPQ, APA', be two sections made by planes perpendicular to BVD, and of which the former is parallel to the base, and is therefore a circle with diameter MQ; and as PN is perpendicular to MQ, we have $PN^2 = MN . NQ$; and the latter will also be a circle, if $\angle AA'V = ABD$; for in that case the triangles AMN, $A'NQ$ are similar, and $\dfrac{NA'}{NQ} = \dfrac{MN}{NA}$;

$$\therefore A'N . NA = MN . QN = PN^2,$$

and as PN is perpendicular to AA', the section APA', which is called a subcontrary section, is a circle; and it is determined by two conditions (1) its plane is perpendicular to the principal section of the cone, and (2) its plane makes the same angle with one of the generating lines of the cone which are in the principal section, as the plane of the base does with the other.

SECTION X.

ON THE GENERAL EQUATION OF CURVES OF THE SECOND ORDER, AND ON CERTAIN GENERAL PROPERTIES OF ALGEBRAICAL CURVES.

Reduction of the General Equation of the Second Order.

222. We shall now proceed to the reduction of the general equation of the second degree

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

where we suppose the co-ordinates rectangular; for if they were oblique, by transforming them to rectangular co-ordinates we should obtain an equation of the same degree as the above, and which could not therefore be more general than the one we have assumed. We shall prove, as affirmed at Art. 62, that this equation by giving a proper position and direction to the origin and axes of the co-ordinates, can always be reduced to one of the forms,

$$Ay^2 + Bx^2 = C,$$

$$y^2 = Ax,$$

the co-ordinates being rectangular; and therefore can never represent any other curve than one of those discussed in the preceding Sections. The principle of the method is to change the system of co-ordinates, without giving any particular values to the quantities which determine the position of the new axes. By that means, indeterminate quantities are introduced into the transformed equation, to which such values can afterwards be assigned as will destroy certain of its terms. Instead of altering both the origin and direction of the co-ordinate axes at once, it is more convenient to effect these changes separately, in the following manner.

223. The general equation of the second order being

$$ay^2 + bxy + cx^2 + dy + ex + f = \phi(x, y) = 0,$$

in order to get rid of the terms involving the simple powers of x and y, we must change the origin without altering the direction of the axes, by putting (Art. 39) $x = x' + h$, $y = y' + k$; this gives

$$ay'^2 + bx'y' + cx'^2 + (2ak + bh + d)y' + (2ch + bk + e)x'$$
$$+ ak^2 + bkh + ch^2 + dk + eh + f = 0,$$

and equating the coefficients of x' and y' to zero, we get

$$\left.\begin{aligned} 2ak + bh + d = 0 \\ 2ch + bk + e = 0 \end{aligned}\right\}\text{.........}\ (1),$$

which give for the co-ordinates of the new origin, provided $b^2 - 4ac$ be different from zero, the single pair of determinate values

$$h = \frac{2ae - bd}{b^2 - 4ac}, \quad k = \frac{2cd - be}{b^2 - 4ac}.$$

Hence the equation becomes, suppressing the accents,

$$ay^2 + bxy + cx^2 + \phi(h, k) = 0,$$

$$\text{where } \phi(h, k) = f + \tfrac{1}{2}(dk + eh) = f + \frac{cd^2 - bed + ae^2}{b^2 - 4ac},$$

as appears by multiplying equations (1) by k and h respectively, and taking their sum; and since this equation remains unaltered when we change x and y into $-x$ and $-y$, the new origin is the centre of the curve.

224. We must now get rid of the term involving the product of the co-ordinates xy, by changing the direction of the axes. For that purpose put (Art. 40),

$$x = x'\cos\theta - y'\sin\theta, \quad y = x'\sin\theta + y'\cos\theta;$$

$$\therefore\ a(x'^2\sin^2\theta + 2x'y'\sin\theta\cos\theta + y'^2\cos^2\theta)$$
$$+ b(x'^2\sin\theta\cos\theta + x'y'\cos^2\theta - x'y'\sin^2\theta - y'^2\cos\theta\sin\theta)$$
$$+ c(x'^2\cos^2\theta - 2x'y'\cos\theta\sin\theta + y'^2\sin^2\theta) + \phi(h, k) = 0,$$

$$\text{or } Ay'^2 + Bx'^2 + \phi(h, k) = 0,$$

$$\left.\begin{aligned} \text{where } A &= a\cos^2\theta - b\cos\theta\sin\theta + c\sin^2\theta \\ B &= a\sin^2\theta + b\cos\theta\sin\theta + c\cos^2\theta \end{aligned}\right\} \dots\dots\dots (2),$$

and the coefficient of $x'y'$

$$= 2a\sin\theta\cos\theta + b(\cos^2\theta - \sin^2\theta) - 2c\cos\theta\sin\theta = 0;$$

which must give a real value for θ, in order that the term involving $x'y'$ may disappear;

$$\therefore (a - c)\sin 2\theta + b\cos 2\theta = 0,$$

$$\text{or } \tan 2\theta = \frac{-b}{a - c}.$$

As the tangent of an angle may have any magnitude, it follows that this equation will always give real values for 2θ; and if we denote by 2α that value of 2θ which lies between zero and π, then the positive values of 2θ are

$$2\alpha, \quad \pi + 2\alpha, \quad 2\pi + 2\alpha, \quad 3\pi + 2\alpha, \quad \&c.;$$

consequently as θ lies between zero and 2π (Art. 41), there are four values of θ, viz.

$$\alpha, \quad \frac{\pi}{2} + \alpha, \quad \pi + \alpha, \quad \frac{3\pi}{2} + \alpha,$$

the two former of which determine two lines at right angles to one another, and the two latter determine the prolongations of these lines; so that if we take one of these lines for the axis of x', the other will be the axis of y'. Hence there exists one system of rectangular axes, and one only, proper to make the product of the co-ordinates $x'y'$ disappear from the transformed equation.

If however we have, at the same time, $b = 0$ and $a = c$, $\tan 2\theta$ becomes indeterminate, or rather the coefficient of $x'y'$ is identically zero; this proves that we may in that case take any two rectangular axes whatever, without introducing the product of the co-ordinates into the transformed equation; and agrees with (Art. 48), for the curve is then a circle.

225. We shall now proceed to the actual determination of the Axes of the curve. Since

$$\tan 2\theta = \frac{-b}{a-c};$$

$$\therefore \cos 2\theta = \frac{a-c}{\sqrt{(a-c)^2+b^2}},$$

$$\sin 2\theta = \cos 2\theta . \tan 2\theta = \frac{-b}{\sqrt{(a-c)^2+b^2}},$$

in these expressions the radical may have either the sign + or −; because we are at liberty to choose either of the new axes for the axis of x; but to avoid all ambiguity, we shall take the radical with a positive sign; then $\sin 2\theta$ will have a sign contrary to that of b.

Hence taking the sum and difference of equations (2), and substituting the above values of $\cos 2\theta$ and $\sin 2\theta$, we get

$$A + B = a + c,$$

$$A - B = (a-c)\cos 2\theta - b\sin 2\theta = \frac{(a-c)^2+b^2}{\sqrt{(a-c)^2+b^2}} = \sqrt{(a-c)^2+b^2};$$

$$\therefore A = \tfrac{1}{2}\left\{a + c + \sqrt{(a+c)^2+m}\right\},$$

$$B = \tfrac{1}{2}\left\{a + c - \sqrt{(a+c)^2+m}\right\},$$

putting $m = b^2 - 4ac$.

226. We have now two cases to consider, according as m is positive or negative.

First let m be negative, then A and B have the same sign; and supposing $\phi(h, k)$ to be of a contrary sign to A and B, and $= -C$, the equation is

$Ay^2 + Bx^2 = C$, which represents an ellipse with semi-axes $\sqrt{\frac{C}{A}}$, $\sqrt{\frac{C}{B}}$, and area $= \frac{-2\pi\phi(h, k)}{\sqrt{4ac-b^2}}$.

If $\phi(h, k) = 0$, the equation is satisfied only by $x = 0$, $y = 0$, i.e. it represents the point which is the origin; and if $\phi(h, k)$ be of the same sign as A and B, the equation can be satisfied by no real values of x and y.

Secondly, let m be positive, then A and B have contrary signs; and whatever be the sign of $\phi(h, k)$ i. e. whether it equals $+ C$ or $- C$, the equation will be of one of the forms

$$\frac{Ay^2}{C} - \frac{Bx^2}{C} = 1, \quad \text{or } \frac{Bx^2}{C} - \frac{Ay^2}{C} = 1,$$

which represents a hyperbola with semi-axes $\sqrt{\frac{C}{A}}$, $\sqrt{\frac{C}{B}}$.

If $C = 0$, the equation is $y = \pm \sqrt{\frac{B}{A}} \, . \, x$, which represents two straight lines through the origin.

227. Next in the equation,

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

let the coefficients be such that $b^2 - 4ac = 0$, and that the numerators in the values of h and k are finite, then the co-ordinates of the centre are infinite, which signifies that the curve has no centre. In this case, as we cannot by changing the origin take away the terms involving the simple powers of x and y, our first object must be to destroy the term involving the rectangle xy. For that purpose put

$$x' \cos\theta - y' \sin\theta \text{ for } x, \text{ and } x' \sin\theta + y' \cos\theta \text{ for } y,$$

and the equation becomes

$$Ay'^2 + Bx'^2 + (d\cos\theta - e\sin\theta)\, y' + (d\sin\theta + e\cos\theta)\, x' + f = 0,$$

the term involving $x'y'$ disappearing, as before, by the condition

$$\tan 2\theta = \frac{-b}{a-c},$$

which gives, since $b^2 = 4ac$,

$$\sin 2\theta = \frac{-b}{\sqrt{(a-c)^2 + b^2}} = \frac{-b}{a+c},$$

$$\cos 2\theta = \frac{a-c}{a+c},$$

taking the radical with the positive sign. Hence by means of the formulæ

$$\cos\theta = \sqrt{\tfrac{1}{2}(1 + \cos 2\theta)},\ \sin\theta = \sqrt{\tfrac{1}{2}(1 - \cos 2\theta)},\ \text{we get}$$

$$d\cos\theta - e\sin\theta = \frac{d\sqrt{a} - e\sqrt{c}}{\sqrt{a+c}} = D,$$

$$d\sin\theta + e\cos\theta = \frac{d\sqrt{c} + e\sqrt{a}}{\sqrt{a+c}} = E.$$

$$\text{Also } A = \tfrac{1}{2}\{a + c + \sqrt{(a-c)^2 + b^2}\} = a + c,$$

$$B = \tfrac{1}{2}\{a + c - \sqrt{(a-c)^2 + b^2}\} = 0,$$

therefore the equation becomes, suppressing the accents,

$$Ay^2 + Dy + Ex + f = 0,$$

$$\text{or } \left(y + \tfrac{1}{2}\frac{D}{A}\right)^2 = \frac{E}{A}\left(-x + \frac{D^2 - 4Af}{4AE}\right),$$

which represents a parabola, latus rectum $= \frac{E}{A}$, and co-ordinates of its vertex $x = \frac{D^2 - 4Af}{4AE}$, $y = -\frac{1}{2}\frac{D}{A}$, and axis parallel to the new axis of x. In this case the co-ordinates of the new origin cannot become infinite; for $A = a + c$ cannot become zero since a and c have the same sign; and if $E = 0$, then the transformed equation will no longer contain x; and being solved with respect to y, it will furnish two constant values for y, so that it will represent two parallel lines.

228. Since the general equation of the second order represents an ellipse, hyperbola, or parabola, according as $b^2 - 4ac$ is negative, positive, or zero, it follows that

$$\left(\frac{y}{k} - 1\right)^2 + \left(\frac{x}{h} - 1\right)^2 \pm \sqrt{\frac{4}{h^2k^2} + m}\,.\,xy = 1,$$

will represent an ellipse, hyperbola, or parabola, according as m is negative, positive, or zero; and under this form

of the equation the axis of x is evidently a tangent to the curve, since when $y = 0$ each value of x becomes equal to h; similarly the axis of y is a tangent.

When $m = 0$, the equation becomes

$$\left(\frac{y}{k} \pm \frac{x}{h}\right)^2 - 2\left(\frac{y}{k} + \frac{x}{h}\right) + 1 = 0;$$

or, taking the upper sign, $\left(\frac{y}{k} + \frac{x}{h} - 1\right)^2 = 0$ representing two straight lines that coincide; with the lower sign we get

$$\left(\frac{y}{k} + \frac{x}{h} - 1\right)^2 = \frac{4xy}{hk},$$

$$\text{or } \sqrt{\frac{y}{k}} \pm \sqrt{\frac{x}{h}} = \pm 1,$$

for the equation to the parabola referred to any two of its tangents as axes. The curve lies wholly between the positive parts of the axes; as long as $x < h$ and $y < k$, the positive sign occurs on both sides; when $x > h$ and $y > k$ the negative sign must be taken on the first side.

229. If in Art. 223 the coefficients of the proposed equation are such that one of the numerators $2ae - bd$ is zero, at the same time that $b^2 = 4ac$, (which two suppositions make the other numerator $2cd - be$ also vanish) both the co-ordinates of the centre become indeterminate; the two equations (1) in that case are equivalent to a single independent equation, and the two lines which they represent, regarding h and k as the co-ordinates, coincide, and there exists an infinite number of centres all situated in that line. The proposed equation, with the above relations among its coefficients, no longer, in fact, represents a curve, but two parallel straight lines; for, solving it, we get

$$y = -\frac{bx + d}{2a} \pm \frac{1}{2a}\sqrt{(b^2 - 4ac)x^2 + 2(bd - 2ae)x + d^2 - 4af},$$

and this in the supposed case becomes

$$y = -\frac{bx+d}{2a} \pm \frac{1}{2a}\sqrt{d^2 - 4af},$$

and therefore represents two parallel straight lines, which are replaced by a single one if $d^2 = 4af$; and become altogether imaginary if $d^2 < 4af$.

230. We shall now shew how to deduce the nature and position of curves of the second order immediately from their general equation, without transformation of co-ordinates.

The value of y in the preceding Art., since the expression under the radical sign has either two real factors or none, may be written (supposing $m = b^2 - 4ac$ to be different from zero) either

$$y = -\frac{bx+d}{2a} \pm \frac{1}{2a}\sqrt{m(x-g)(x-h)}, \quad (1)$$

$$\text{or } y = -\frac{bx+d}{2a} \pm \frac{1}{2a}\sqrt{m\{(x-\alpha)^2 + \beta^2\}}. \quad (2)$$

The line $y = -\dfrac{bx+d}{2a}$ is evidently a diameter of the curve, for it bisects all chords parallel to the axis of y. If m be negative, the value of y in the former equation is real only from $x = g$ to $x = h$, supposing $h > g$, and cannot become infinite between those limits; and in the latter y is imaginary for every value of x; therefore the curve is limited in all directions, and is an ellipse situated as in fig. 84, where $NN' = h - g$; and $P'N'$, PN are the first and last ordinates touching the ellipse at the extremities of the diameter PP' whose equation is $2ay + bx + d = 0$. If M be the middle point of NN', then when $x = OM$, the irrational part of y, represented by DC, attains its greatest value; therefore the tangent at D is parallel to PP'; hence CP, CD are a pair of semi-conjugate diameters, whose magnitudes and inclination being known, the axes of the ellipse may be determined (Art. 145). Also, w being the angle be-

tween the axes, the area of the ellipse $= \pi . CD . CP \sin PCD$

$$= \pi \sin w . CD . MN = \frac{\pi \sin w \sqrt{-m}}{8a} (h - g)^2$$

$$= \frac{\pi \sin w}{2a(-m)^{\frac{3}{2}}} \{(2ae - bd)^2 + m(4af - d^2)\}.$$

If $h = g$, then $x = g$ is the only value that makes y real, and the ellipse is reduced to a point.

231. When m is positive, x may have in (1) all values except those lying between g and h, and in (2) all values whatever; therefore, in both cases, the curve goes off to infinity in four directions, and is a Hyperbola.

For equation (1), the curve (fig. 85) is met by the diameter $2ay + bx + d = 0$ in the points P, P', for which $ON' = g$, $ON = h$; and no part of the curve lies between the parallels PN, $P'N'$. Also the equation to the asymptotes (obtained by developing the irrational part of y and neglecting negative powers of x) is

$$y = -\frac{bx + d}{2a} \pm \frac{\sqrt{m}}{2a} \{x - \tfrac{1}{2}(g + h)\};$$

both of which lines meet the diameter TC in C the middle point of PP'. If $h = g$, the curve is reduced to two straight lines coincident with the asymptotes. For equation (2) the hyperbola is situated as in fig. 86, each branch being convex towards the diameter TC which does not meet the curve at all. When $x = \alpha = OM$, the irrational part of y receives its least value represented by CD, and the tangent at D is consequently parallel to the diameter TC. Also the equation to the asymptotes is

$$y = -\frac{bx + d}{2a} \pm \frac{\sqrt{m}}{2a} (x - \alpha),$$

both of which meet the diameter TC in C the middle point of DD'. When $\beta = 0$, the curve is reduced to two straight lines coincident with the asymptotes.

232. If in the general equation $c = 0$, then m is positive and the curve is a hyperbola; therefore if $a = 0$ (in which case the preceding results seem to fail) solving the equation with respect to x, our conclusions would still hold. In the case however of the square of one of the variables being wanting, the simpler plan is to solve the equation with respect to that variable, and we get

$$y(bx + d) + cx^2 + ex + f = 0,$$

or, by division,

$$y = px + q + \frac{f}{bx + d},$$

corresponding to which equation the position of the curve is that in fig. 87, one of the asymptotes being CP with equation $y = px + q$, and the other CD parallel to the axis of y, whose equation is $bx + d = 0$. If $f = 0$, the equation is reduced to $(px + q - y)(bx + d) = 0$, representing two straight lines.

233. When $m = 0$, the general equation, solved with respect to y, becomes

$$y = -\frac{bx + d}{2a} \pm \frac{1}{2a}\sqrt{2px + q};$$

then if p be positive, x may be taken from zero to infinity, and y at the same time increases to infinity, both positively and negatively; but if x be taken negative beyond a certain limit, y becomes imaginary; therefore the curve has only two infinite branches, and is a parabola in the position QPR represented by fig. 88, TP being the diameter meeting the curve in P. If p be negative, x must be taken negatively to infinity, and the curve has the reversed position $Q'PR'$. If $p = 0$, the equation represents two straight lines parallel to the diameter TP and at equal distances from it; which coincide if $q = 0$, and become imaginary if q be negative.

234. In determining the actual position and magnitude of the Axes of a conic section from its equation, it will be

always found convenient, as a first step, to transfer the origin to the centre, when it exists; or to a point in the curve, when there is no centre. The following are instances of the principal cases that can occur.

Ex. 1. $y^2 - 2mxy + (m^2 + n^2)x^2 - n^2c^2 = 0,$

which represents an ellipse (fig. 84);

$$\therefore\ y = mx \pm n\sqrt{c^2 - x^2},$$

$$\text{when } x = 0,\ y = CD = nc,$$

$$\text{when } x = CQ = c,\ y = PQ = mc;$$

$$\therefore\ a^2 + b^2 = CP^2 + CD^2 = c^2(1 + m^2 + n^2),$$

$$ab = CD \times CQ = nc^2,$$

$$\text{and } a^2 - (a^2 - b^2)\sin^2\phi = c^2, \text{ where } \angle ACQ = \phi;$$

which equations give the magnitude and position of the Axes.

Ex. 2. $y^2 - 2mxy + (m^2 - n^2)x^2 + n^2c^2 = 0,$

which represents a hyperbola (fig. 85);

$$\therefore\ y = mx \pm n\sqrt{x^2 - c^2},$$

$$\text{when } x = 0,\ y = CD\sqrt{-1} = nc\sqrt{-1},$$

$$\text{when } x = CQ = c,\ y = PQ = mc;$$

$$\therefore\ a^2 - b^2 = CP^2 - CD^2 = c^2(1 + m^2 - n^2),$$

$$ab = CD \times CQ = nc^2,$$

$$\text{and } a^2 - (a^2 + b^2)\sin^2\phi = c^2, \text{ where } \angle ACQ = \phi;$$

which equations give the magnitude and position of the Axes.

Ex. 3. $y^2 - (m + m')xy + mm'x^2 - c^2 = 0,$

which represents a hyperbola (fig. 86);

$$\therefore\ y = \tfrac{1}{2}(m + m')x \pm \sqrt{\tfrac{1}{4}(m' - m)^2x^2 + c^2}.$$

The diameter CP whose equation is $y = \frac{1}{2}(m + m')x = \tan\alpha . x$, falls between the two branches of the curve, and $y = m'x$,

$y = mx$, are the equations to the asymptotes CL', CL, so that if CB be the conjugate axis inclined to the axis of x at an angle β,

$$\beta = \tfrac{1}{2}(L'Cx + LCx) = \tfrac{1}{2}(\gamma' + \gamma) \text{ suppose};$$

$$\therefore \cot\beta = \frac{1}{\sin 2BCx} + \frac{1}{\tan 2BCx} = \frac{\sqrt{1+m'^2}\sqrt{1+m^2}+1-mm'}{m+m'};$$

and if the tangent at D, which is parallel to CP, meet the transverse axis in t, we get a and b in terms of m, m', from

$$a^2 = Ct \,.\, CD \cos BCx = \frac{c^2}{1 + \tan\alpha\tan\beta}, \text{ and } \frac{b}{a} = \cot BCL,$$

$$\text{or } a^2\{1 + \cos(\gamma' - \gamma)\} = b^2\{1 - \cos(\gamma' - \gamma)\} = c^2\cos\gamma'\cos\gamma.$$

Ex. 4. $y = mx + \dfrac{c}{x}$ which represents a hyperbola (fig. 87) one of whose asymptotes is the axis of y, and the other the line CP with equation $y = mx$. Let CA be the transverse axis,

$$\therefore \tan ACx = \tan\tfrac{1}{2}(90^0 + PCx) = \tan PCx + \sec PCx$$
$$= \sqrt{1 + m^2} + m.$$

Let x' and y' be co-ordinates of A; then

$$mx' + \frac{c}{x'} = x'\sqrt{1 + m^2} + mx'; \;\therefore x'^2 = \frac{c}{\sqrt{1 + m^2}};$$

$$\therefore a^2 = 2c(\sqrt{1 + m^2} + m),$$

$$\text{and } \frac{b}{a} = \cot ACx; \;\therefore b^2 = 2c(\sqrt{1 + m^2} - m).$$

Ex. 5. $$y^2 - 2myx + m^2x^2 - cx = 0,$$

which represents a parabola (fig. 88);

$$\therefore y = mx + \sqrt{cx}.$$

The axis of y is a tangent at P the origin, and the line PV, whose equation is $y = mx$, a diameter.

Let $\angle yPV = \alpha$, $PN = x$, then $PV = \dfrac{x}{\sin\alpha}$ and $QV = \sqrt{cx}$;

$$\therefore cx = \frac{x}{\sin\alpha} \times \frac{4a}{\sin^2\alpha}; \therefore \text{latus rectum } (4a) = c\sin^3\alpha$$

$$= \frac{c}{(1+m^2)^{\frac{3}{2}}}; \text{ and distance of } P \text{ from the axis} = \frac{2a}{m}.$$

General Properties of Algebraical Curves.

235. The general equation of the n^{th} degree between x and y, ought to contain all the combinations of the powers of x and y in which the sum of the indices does not exceed n; therefore when complete and arranged according to descending powers of y, it will be

$$a_0y^n + (b_0 + b_1x)\,y^{n-1} + (c_0 + c_1x + c_2x^2)\,y^{n-2} + \&c.$$
$$+ (l_0 + l_1x + l_2x^2 + \&c. + l_nx^n) = 0.$$

All equations between two variables x and y, which can be reduced to this form, are called algebraical, all others are called transcendental; hence arises the distinction of lines into algebraical and transcendental, according as their equations are algebraical or transcendental.

236. The classification of lines in different orders, according to the degrees of their equations, would be to little purpose, if by changing the axes of the co-ordinates we altered the degree of the equation. But this is not the case. For, having given, between x and y, the equation to a line referred to certain axes, in order to get the equation to the same line referred to new axes, we must replace x and y in the given equation by the values found in Art. 41; and as these values are of the first degree in x' and y', it follows that the degree of the equation cannot be raised by this substitution. Neither can the degree of the transformed equation be less than that of the primitive equation; for if it could, then, by what has been proved, we could not return from it to the primitive equation, which is absurd.

237. The general equation of any degree comprehends not only all lines of the order expressed by that degree, but also

all lines of inferior orders. Thus the above general equation of the n^{th} degree, by making $a_0 = b_1 = c_2 = \ldots = l_n = 0$, degenerates into the equation of the $(n-1)^{\text{th}}$. Also the equation of the second degree

$$(y - mx - c)(y - m'x - c') = 0$$

is clearly verified either by putting $y = mx + c$, or $y = m'x + c'$, which represent two lines of the first order; so that the proposed equation does not in reality represent a line of the second order at all, but two straight lines; or only one even of these, if $m = m'$, $c = c'$. Similarly, the equation of the third order

$$(y - mx - c)(ay - x^2) = 0$$

represents a line of the first order, and one of the second whose equation is $ay - x^2 = 0$. And, in general, according as a proposed equation of any degree is not, or is capable of being resolved into factors which are rational with respect to the variables x and y, it will represent a single line of the corresponding order, or several distinct lines of inferior orders.

238. A straight line cannot meet a curve of the n^{th} order in more than n points.

Let the co-ordinates be transformed so that the proposed line may be the axis of x, and let $V = 0$ be the resulting equation to the curve; in order to determine the points in which it is intersected by the straight line, we must put $y = 0$ in the equation $V = 0$, and the corresponding values of x will be the abscissæ of the required points. But $V = 0$ being of the n^{th} degree, the equation for determining x will be at the most of the n^{th} degree; therefore x cannot have more than n values, and there cannot be more than n points of intersection; but there may be fewer than n, for the equation for determining x may be of a degree inferior to n, and may have equal or imaginary roots.

239. The general equation of the n^{th} degree between two variables, when complete, contains $1 + 2 + 3 + \&c. + (n + 1)$, or $\frac{1}{2}(n + 1)(n + 2)$, arbitrary constants, in which, since we

may divide the whole equation by one of them, there is one superfluous which might be suppressed; consequently the number of independent constants is

$$\tfrac{1}{2}(n+1)(n+2)-1, \quad \text{or } \tfrac{1}{2}n(n+3).$$

Hence a curve of the n^{th} order may be made to fulfil $\frac{1}{2}n(n+3)$ conditions; as, for instance, to pass through $\frac{1}{2}n(n+3)$ points; for, giving to x and y their values at each of the given points, we get $\frac{1}{2}n(n+3)$ different equations by means of which the values of the constants may be determined. Hence a curve of the second order may be determined so as to pass through five given points; as will be seen in the following Problem.

240. To determine the conic section which shall pass through five given points.

Take the axes of the co-ordinates so that each axis contains two of the given points; and let y_1, y_2, be the ordinates of the points situated in the axis of y; x_1, x_2, the abscissæ of the points situated in the axis of x; and x_3, y_3, the co-ordinates of the fifth given point. Then substituting successively the co-ordinates of each of these points in the place of x and y in the general equation (where every coefficient is divided by the constant term),

$$ay^2 + bxy + cx^2 + dy + ex + 1 = 0,$$

we get the five equations

$$ay_1^2 + dy_1 + 1 = 0, \quad ay_2^2 + dy_2 + 1 = 0,$$
$$cx_1^2 + ex_1 + 1 = 0, \quad cx_2^2 + ex_2 + 1 = 0,$$
$$ay_3^2 + bx_3y_3 + cx_3^2 + dy_3 + ex_3 + 1 = 0,$$

which give for the five unknown quantities, the values

$$a = \frac{1}{y_1y_2}, \quad d = -\frac{y_1+y_2}{y_1y_2}, \quad c = \frac{1}{x_1x_2}, \quad e = -\frac{x_1+x_2}{x_1x_2},$$

$$b = -\frac{1}{x_3y_3}\left\{\frac{y_3(y_3-y_2-y_1)}{y_1y_2} + \frac{x_3(x_3-x_2-x_1)}{x_1x_2} + 1\right\}.$$

Now provided no three of the given points be in a straight line, none of the quantities x_1, x_2, &c. is zero; therefore the above values of a, b, &c. are neither infinite, nor indeterminate, and none of them has more than one value; therefore through five points, provided no three be in a straight line, a conic section, and only one, can be made to pass.

241. Every curve of the n^{th} order which passes through $\frac{1}{2}n(n+3)-1$ given fixed points, will also pass through $\frac{1}{2}n(n-3)+1$ additional fixed points.

Since the given points are one fewer in number than what would be sufficient to completely determine a curve of the n^{th} order, an infinite number of such curves may be described through them. Of these, let us consider any two whose equations are $M=0$, $M'=0$; then the equation $M'+\mu M=0$, (1) (where μ is an indeterminate constant) will include all the curves of the n^{th} order that can pass through the given points, since the equation of every such curve could involve only one undetermined constant. But equation (1) will be satisfied by every pair of values of x and y which satisfy $M=0$, $M'=0$; therefore the curve (1) will pass through the n^2 points of intersection of $M=0$, $M'=0$; that is, all the curves will pass through the points of intersection of any two of them; therefore all the points of intersection must be fixed points; and the $\dfrac{n(n+3)}{2}-1$ given points will determine the remaining points of intersection. Hence every curve of the n^{th} order, besides passing through a number of fixed points one less than the number sufficient to completely determine it, will also pass through an additional number of fixed points such that added to the former it makes up n^2, the entire number of points in which two curves of the n^{th} order can intersect one another.

Hence 8 given points of a curve of the third order will determine a ninth point of the same curve; and 13 given points of a curve of the 4th order will determine 3 new points of the same curve.

242. To find the position of the centre of any curve.

The centre of a curve is a point C (fig. 24), such that any chord of the curve PP' drawn through it, is bisected in it. (It must be observed, however, that if PP' meet the curve in more points than two, it is sufficient that these points combined in a certain order should be two and two equally distant from C.) If the curve be referred to any two axes originating in C, and PN, $P'N'$ be the ordinates parallel to Cy of the extremities of a chord, we see from the equal triangles PCN, $P'CN'$, that these ordinates are equal and of contrary signs; the same thing is true for the abscissæ of P and P'; as well as for the extremities of every other chord passing through C. If therefore $\phi(x, y) = 0$ be the equation to the curve, and if it be satisfied by $x = a$, $y = b$, it must also be satisfied by $x = -a$, $y = -b$; that is, it must be such as not to alter when the signs of the two variables are changed; and conversely, if it have this property, the origin is the centre of the curve. When $\phi(x, y) = 0$ is algebraic, it cannot have the above property unless the dimension of every term be even in an equation of an even degree, and odd in an equation of an odd degree; for in the former case the equation is not at all altered by replacing x and y by $-x$ and $-y$; and in the latter (in which case the equation cannot have a constant term) the sign of every term will be altered, and therefore the whole equation unaltered. Hence to find whether a proposed curve admits of a centre, we must refer it to parallel axes through a new origin having co-ordinates h, k, by putting $x = x' + h$, $y = y' + k$; and equate to zero the coefficients of all the terms which are of a dimension different (as far as regards odd and even) from the degree of the equation; if these conditions can all be satisfied by real and finite values of h and k, the curve has a centre, and h and k are its co-ordinates; in the contrary case the curve has no centre. Of this process we have an example at Art. 223.

243. The locus of the middle points of a system of parallel chords of any curve, is called its diametral curve.

If the curve be of the n^{th} order, the points of intersection with its ordinates real or imaginary will be in number n; and their combinations on the same indefinite line will form $\frac{1}{2}n(n-1)$ different chords, and as many middle points, and therefore the diametral curve, since it may be met by an indefinite line in $\frac{1}{2}n(n-1)$ points, will have an equation of the degree $\frac{1}{2}n(n-1)$. For curves of the second order, since $n=2$, the diametral curves can only be straight lines; for curves of the third order, the diametral curves are also of the third order.

244. To find the locus of the middle points of a system of parallel chords of any curve.

Let the chords be parallel to a line through the origin whose equation is $y = mx$, and let $\phi(x, y) = 0$, be the equation to the curve; also let x', y', be the co-ordinates of the middle point of any one of these chords, and take it for the origin without altering the direction of the axes, and therefore put $x' + x$ for x, and $y' + y$ for y; then the transformed equation to the curve is $\phi(x' + x, y' + y) = 0$, and the equation to the chord is $y = mx$. Hence the values of x, corresponding to the points of intersection of the curve and chord, result from the equation $\phi(x' + x, y' + mx) = 0$, or

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \&c. + p_n = 0, \text{ suppose};$$

and because the origin bisects the chord, this equation must be satisfied by $-x$,

$$\therefore\ x^n - p_1 x^{n-1} + p_2 x^{n-2} - \&c. + (-1)^n p_n = 0;$$

between which two equations if we eliminate x, we obtain a relation between x' and y', which is the equation to the required locus.

245. Thus if $\phi(x, y) = 0$ be the general equation of the second order,

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

putting $x = x' + x$, and $y = y' + mx$, we get

$$a(y' + mx)^2 + b(x + x')(y' + mx) + c(x + x')^2$$
$$+ d(y' + mx) + e(x + x') + f = 0,$$

and because the values of x are to be equal and of contrary signs, the term involving the first power of x must disappear;

$$\therefore 2amy' + b(x'm + y') + 2cx' + dm + e = 0,$$

$$\text{or } y'(2am + b) + x'(2c + bm) + dm + e = 0,$$

the equation to a straight line. Hence there will be an infinite number of diameters corresponding to the various values of m.

If the diameter is to be perpendicular to its chords, we must have

$$1 + m\left(-\frac{2c + bm}{2am + b}\right) = 0, \quad \text{or } m^2 + \frac{2(c - a)}{b}m - 1 = 0,$$

which will necessarily give two real values of m; hence there are generally two diameters which bisect their ordinates perpendicularly.

If $\phi(x, y) = 0$ be the general equation to curves of the third order, and

$$\phi(x' + x,\ y' + mx) = x^3 + p_1x^2 + p_2x + p_3 = 0,$$

$$\text{then also } x^3 - p_1x^2 + p_2x - p_3 = 0;$$

therefore, adding and subtracting, $x^2 + p_2 = 0$, $p_1x^2 + p_3 = 0$;

$\therefore\ p_3 = p_1p_2$ is the equation to the diametral curve.

246. Not only are Algebraical curves distributed into orders according to the degree of their equations; but also the different families of lines are investigated, which may be comprised amongst those of the same order; and even the different species of each family, if necessary. The individual lines of the same family, or species, are then classified according to certain characteristics easy to be recognized, which completely distinguish them from one another; and lastly it is

endeavoured to determine the form and properties of each of them. This has been here effected for equations of the first and second degrees; the former gives only straight lines, as has been said; the latter gives three species of curves sufficiently distinct; viz. the parabola, which has no centre; and the ellipse and hyperbola, both of which have a centre, but only the latter has asymptotes.

The enumeration of lines of the third order was first made by Newton, who found 72 species comprized in 14 divisions; Stirling added 4 species which had been omitted; and, lastly, Cramer added two more, making in all 78 species.

On tracing Curves from their Equations.

247. When a curve passes through the origin, the angle at which it cuts the axis of x may be determined by taking the limit of $\frac{y}{x}$ when $x = 0$, which will be the value of the tangent of that angle.

Let AP be a curve passing through the origin A (fig. 72), P a point in it near A with co-ordinates $AN = x$, $NP = y$; draw the secant AP, then $\tan PAN = \frac{y}{x}$; now let P move up to and coincide with A, then the secant AP coincides with AT the tangent to the curve at A, and

$$\tan TAN = \text{limit of } \tan PAN = \text{limit of } \frac{y}{x}, \text{ when } x = 0.$$

Hence the angles at which a proposed curve cuts the co-ordinate axes may always be determined; for we have only to transfer the origin to one of the points in question, and in the transformed equation take the limit of $\frac{y}{x}$ by putting $x = 0$.

248. In tracing curves from their equations, whenever y is given, or can be found, in an explicit function of x, it will be best to use algebraical processes alone.

First, determine the points where the curve cuts the axis of x, and its shape at those points. For that purpose transfer the origin of co-ordinates, if necessary, to one of the points; expand y in a series ascending by powers of x, and let

$$y = ax^m + bx^n + \&c.$$

If $m < 1$, limit of $\frac{y}{x} = \infty$; therefore the curve is perpendicular to the axis of x, and immediately afterwards is concave towards that axis.

If $m = 1$, limit of $\frac{y}{x} = a$; therefore the curve cuts the axis of x at an angle whose tangent is a, and immediately afterwards is situated above or below the tangent, i.e. is convex or concave towards the axis of x, according as b is positive or negative.

If $m > 1$, limit of $\frac{y}{x} = 0$; therefore the curve touches the axis of x, and immediately afterwards is convex towards that axis.

Similarly, the form of the curve at all its other intersections with both the axes may be found.

Secondly, determine the nature of the infinite branches; and to that end expand y in a descending series of powers of x (on the supposition that both x and y are very great), and let

$$y = ax^m + bx^n + \ldots\ldots + ex + f + \frac{g}{x} + \ldots\ldots;$$

$\therefore\ y = ax^m + bx^n + \ldots\ldots + ex + f$ is the equation to the asymptotic curve, above or below which the given curve is situated, according as g is positive or negative.

If $m = 1$, the equation to the asymptote is $y = ex + f$, representing a straight line; and the curve is situated above

or below the asymptote, according as g is positive or negative; consequently, as the curve will be convex towards the rectilinear asymptote with which it continually tends to coincide, it will ultimately be convex or concave towards the axis of x, according as the first of the neglected terms is positive or negative.

In this case the infinite branch represented by

$$y = ex + f + \frac{g}{x} + \ldots\ldots$$

is said to be hyperbolic.

If $m > 1$, the infinite branch is parabolic; and it is concave or convex towards the axis of x, according as its asymptote is concave or convex towards that axis.

249. Having thus found the figure of the curve at the points where it cuts the axes, and also when x and y are very great, the intermediate parts may generally be traced. For the actual position of the maximum or minimum ordinates and points of contrary flexure, recourse must be had to the methods of the Differential Calculus.

If the equation to a curve can be resolved, and give for y the values V, V', &c. functions of x and constants, we must trace separately each of the curves represented by $y = V$, $y = V'$, &c., all of which will be particular branches of the proposed curve. The branches cannot terminate abruptly; they will either go on to infinity, or the ordinates will become imaginary, in which case two branches will be united and mutually continue one another.

Ex. 1. To trace the curve whose equation is $ay^3 = (a^2 - x^2)^2$ (fig. 73).

Since the equation does not alter when $-x$ is written for x, the curve is symmetrical with respect to the axis of y; also for any value of x, y has only one possible value, and is always positive. When $x = 0$, $y = a$, and as

x increases either way, y diminishes; therefore a is a maximum value of y, and the curve cuts the axis of y at B at right angles, and is concave to the axis of x. When $x = a$, $y = 0$, and the curve cuts the axis of x at right angles at A, because if we remove the origin to that point by making $x = a + x'$, we get $ay^3 = (2ax' + x'^2)^2$, and therefore limit of $\dfrac{y^3}{x'^3} =$ limit of $\dfrac{(2a + x')^2}{ax'} = \infty$. When $x > a$, the equation becomes $ay^3 = (x^2 - a^2)^2$, and as x increases, y increases, till x is very large, when the relation between them approximates to

$$a^{\frac{1}{3}}y = x^{\frac{4}{3}}\left(1 - \frac{a^2}{x^2}\right)^{\frac{2}{3}} = x^{\frac{4}{3}}\left(1 - \frac{2}{3}\frac{a^2}{x^2}\right) = x^{\frac{4}{3}} - \frac{2}{3}\frac{a^2}{x^{\frac{2}{3}}};$$

$\therefore\ a^{\frac{1}{3}}y = x^{\frac{4}{3}}$ is the equation to the asymptotic parabola ZOZ', below which the curve lies, because the second term of the expansion of y is negative. Hence the figure of the curve is that annexed, having a point of inflexion at P; for the curve is concave to the axis of x at A, and afterwards convex because the parabola with which it tends to coincide is so. There is of course another point of inflexion at P'.

Ex. 2. To trace the curve whose equation is

$$y = \frac{x - 3}{(x - 1)(x - 2)} \text{ (fig. 74).}$$

When $x = 0$, $y = -\dfrac{3}{2}$, and the curve cuts the axis of y at D; as long as $x < 1$, y is negative and becomes infinite when $x = 1$; therefore the ordinate BE corresponding to $x = 1$ is an asymptote, and we thus get the branch DE.

When x is between 1 and 2, y is positive, and becomes very great both when x is a little less than 2 and a little greater than 1, and gives the portion $E'GF'$.

When $x = 2$, y is infinite, so that the ordinate FF' is an asymptote.

When x lies between 2 and 3, y is negative and gives the portion FH.

When $x = 3$, $y = 0$, and the curve cuts the axis at H.

When $x > 3$, y is positive and gives the portion HK; and when x is very great,

$$y = \frac{x}{x^2 - 3x} = \frac{1}{x - 3} = 0;$$

therefore the axis of x is an asymptote.

When x is negative, the equation is

$$y = -\frac{x + 3}{(x + 1)(x + 2)};$$

therefore y is always negative, and diminishes as x increases, and becomes 0 when $x = \infty$ and gives the branch DL.

Ex. 3. To trace the curve whose equation is

$$ay = x^2 + x\sqrt{2ax - x^2} \text{ (fig. 75).}$$

Taking the radical, which of course admits of a double sign, first with a positive sign, we have when $x = 0$, $y = 0$, and limit of $\frac{y}{x} = 0$; therefore the curve passes through the origin A and touches the axis of x; when $x = a$, $y = 2a$, and when $x = 2a$, $y = 4a$, beyond which y is impossible; we thus get the portion of the curve AED, the ordinate CD being a tangent at D. Again taking the radical with a negative sign, so that

$$ay = x^2 - x\sqrt{2ax - x^2},$$

$x = 0$ gives $y = 0$, and limit of $\frac{y}{x} = 0$, as before; as long as $x < a$, y is negative; when $x = a$, $y = 0$, and putting $x' + a$ for x we get, supposing x' very small,

$$ay = ax' + \frac{3}{2}x'^2; \quad \therefore \text{ limit of } \frac{y}{x'} = 1;$$

hence the portion of the curve AFB touches the axis of x at A, and cuts it at an angle of 45^0 at B, and is there situated above the tangent. When $x = 2a$, $y = 4a = DC$, and for greater values of x, y is impossible; hence DC is a tangent to both portions of the curve at D, the point in which they are united. For all negative values of x, y is impossible. The curve is therefore such as is represented, the tangent being parallel to the axis of x at F and E.

PROBLEMS.

1. The equation $y^2 - 2xy \sec\alpha + x^2 = 0$, represents two straight lines that include an angle $= \alpha$.

Solving the equation we get

$$y^2 - 2xy \sec\alpha + x^2 \sec^2\alpha = x^2 \tan^2\alpha;$$

$$\therefore\ y = x(\sec\alpha \pm \tan\alpha),$$

which represents two straight lines; and if θ be the angle between them,

$$\tan\theta = \frac{m - m'}{1 + mm'}, = \frac{2\tan\alpha}{2} = \tan\alpha.$$

2. To find the equation to a straight line which shall pass through the point of intersection of two given lines and bisect the angle between them.

Let CB, CB' (fig. 13) be the two given lines, determined by the equations $y = mx + c$, $y = m'x + c'$, and making the angles α, α', with the axis of x; CE the required line making an angle θ with that axis; then the co-ordinates of C are known from Art. 24;

$$\text{and } \theta = \alpha' + \frac{\alpha - \alpha'}{2} = \frac{\alpha + \alpha'}{2};$$

$$\therefore\ \cot\theta = \cot\frac{\alpha + \alpha'}{2} = \frac{1}{\sin(\alpha + \alpha')} + \frac{1}{\tan(\alpha + \alpha')}$$

$$= \frac{\sqrt{1 + m^2}\,.\sqrt{1 + m'^2}}{m + m'} + \frac{1 - mm'}{m + m'};\ \text{(Art. 25)}$$

$$\therefore\ y - \frac{m'c - mc'}{m' - m} = \frac{m + m'}{\sqrt{1 + m^2}\,.\sqrt{1 + m'^2} + 1 - mm'}\,.\left(x - \frac{c - c'}{m' - m}\right),$$

the required equation; the negative sign of the radical referring to CE' which is perpendicular to CE.

3. To find the equation to the line joining the middle points of the diagonals of a quadrilateral.

Let $y = mx$, $y = nx$, be the equations to the sides AB, AD, (fig. 89); and $y = k$, $x = h$, the equations to the other two sides to which the co-ordinate axes are drawn parallel. Then if O, O' be the middle points of the diagonals AC, BD, the co-ordinates of O are $\frac{1}{2}k$, $\frac{1}{2}h$ and of O', $\frac{1}{2}(k + nh)$, $\frac{1}{2}\left(h + \frac{1}{m}k\right)$; therefore the equation required to OO' is

$$y - \tfrac{1}{2}k = \frac{mnh}{k}(x - \tfrac{1}{2}h).$$

It may be remarked that if we produce AB, DC, two opposite sides to meet in M; and AD, BC, the other two sides to meet in N; then MN may be regarded as a third diagonal of the quadrilateral, since it joins the points of intersection of the sides; and the co-ordinates of the middle point of MN will be

$$x = \tfrac{1}{2}\left(h + \frac{1}{n}k\right), \quad y = \tfrac{1}{2}(k + mh);$$

which evidently fulfil the equation to OO'; therefore MN is bisected by OO' produced.

4. The line bisecting the diagonals of any quadrilateral, passes through the intersection of the lines joining the middle points of the opposite sides.

The co-ordinates of the middle point of AD (fig. 89) are $x = \frac{1}{2}h$, $y = \frac{1}{2}nh$, and of BC $y = k$, $x = \frac{1}{2}\left(h + \frac{k}{m}\right)$; therefore the line joining these middle points has for its equation

$$y - \tfrac{1}{2}nh = \frac{2mk - mnh}{k}(x - \tfrac{1}{2}h) \quad (1);$$

but equation to OO' is

$$y - \tfrac{1}{2}k = \frac{mnh}{k}(x - \tfrac{1}{2}h);$$

therefore for point of intersection of (1) and OO',

$$\tfrac{1}{2}(k - nh) = \frac{2m(k - nh)}{k}(x - \tfrac{1}{2}h), \text{ or } x - \tfrac{1}{2}h = \frac{k}{4m}.$$

But the line joining the middle points of AB and DC has for its equation

$$y - \tfrac{1}{2}k = \frac{mnh}{2mh - k}\left(x - \frac{k}{2m}\right) \quad (2);$$

and consequently for intersection of (2) with OO' we have

$$x - \frac{k}{2m} = \frac{2mh - k}{k}(x - \tfrac{1}{2}h), \text{ which also gives } x - \tfrac{1}{2}h = \frac{k}{4m};$$

therefore (1) and (2) both intersect OO' in the same point.

5. If any four points in a plane, A, B, C, D be joined by straight lines two and two in every possible way, the joining lines being produced if necessary to intersect; and if these points of intersection be also joined two and two in every possible way; every straight line in the figure so formed will be harmonically divided.

Take HM, HN (fig. 89) for the co-ordinate axes, and let $HD = a$, $HA = a'$, $HN = h$, $HF = b$, $HG = k$, $HM = c$.

Then the equations to BD, AC, are respectively (Art. 24)

$$-\frac{x}{a'} + \frac{y}{c} - 1 = n\left(\frac{x}{h} + \frac{y}{k} - 1\right),$$

$$\frac{x}{a} + \frac{y}{c} - 1 = n\left(\frac{x}{h} + \frac{y}{k} - 1\right),$$

both of which are satisfied by $x = 0$, $y = b$;

$$\therefore \frac{b}{c} - 1 = n\left(\frac{b}{k} - 1\right).$$

Also since BD, AC, pass respectively through the points D, A,

$$\frac{a}{a'} + 1 = n\left(-\frac{a}{h} + 1\right), \quad \frac{a'}{a} + 1 = n\left(\frac{a'}{h} + 1\right);$$

$$\therefore \frac{2}{h} = \frac{1}{a} - \frac{1}{a'}, \text{ or } \frac{2}{a+a'} = \frac{1}{a'} + \frac{1}{a'+h};$$

which shews that AN is harmonically divided.

Again,

$$\frac{k}{c} \cdot \frac{b-c}{b-k} = n = \frac{h}{a} \cdot \frac{a'+a}{a'+h},$$

$$\therefore \frac{b}{c} \cdot \frac{k-c}{b-k} = \frac{a'}{a} \frac{h-a}{h+a'} = 1, \text{ or } \frac{2}{k} = \frac{1}{b} + \frac{1}{c},$$

which shews that HM is harmonically divided; and the same may be similarly proved of the other lines of the figure.

6. To express the area of a triangle in terms of the co-ordinates x, y; x', y'; x'', y''; of its angular points.

Let P, P', P'' be its angular points, through which draw ordinates meeting the axis of x in N, N', N''; then (Art. 35)

$$\begin{aligned} \text{area} &= \text{trapez. } NPP'N' + N'P'P''N'' - NPP''N'' \\ &= \tfrac{1}{2}\sin\omega\{(y'+y)(x'-x)+(y''+y')(x''-x')-(y''+y)(x''-x)\} \\ &= \tfrac{1}{2}\sin\omega\{(y'+y)(x'-x)+(y''+y')(x''-x')+(y+y'')(x-x'')\}. \end{aligned}$$

7. If through any point in a line joining the vertex of a triangle with the middle point of the base, lines be drawn from the extremities of the base to meet the other two sides or those sides produced, the line joining the points of intersection will be parallel to the base.

Take the base AB (fig. 90) for the axis of x, and the line bisecting it CD for that of y; and let $OA = OB = a$, $OD = b$, $OC = c$; then the equations to BC, AD, are

$$\frac{x}{a} - \frac{y}{c} = 1, \quad -\frac{x}{a} + \frac{y}{b} = 1;$$

therefore for the ordinate of E, we get

$$-\frac{y}{c} + \frac{y}{b} = 2.$$

The equations to BD, AC, are

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{a} + \frac{y}{c} = -1,$$

$$\therefore \text{ for } F, \quad -\frac{y}{c} + \frac{y}{b} = 2;$$

which, being the same as for E, is the equation to the line passing through those points; that line is therefore parallel to AB.

8. A perpendicular being dropped from a point (a, b) upon a line whose equation, referred to axes inclined at an angle ω, is

$$\frac{x}{a + b\cos\omega} + \frac{y}{b + a\cos\omega} = 1,$$

to find its equation and its length.

Its equation will be (Art. 33)

$$(y - b)(m + \cos\omega) + (x - a)(1 + m\cos\omega) = 0,$$

or, substituting for m its value $-\dfrac{b + a\cos\omega}{a + b\cos\omega}$, and reducing,

$$(y - b)\,b = (x - a)\,a.$$

Also, substituting for m its value in the formula of Art. 34, we get for its length

$$p = \frac{ab\sin^2\omega}{\sqrt{a^2 + b^2 + 2ab\cos\omega}}.$$

9. To find the polar equation to a line which shall pass through two points whose polar co-ordinates are given.

Let P', P'', (fig. 14) be the two given points respectively determined by the polar co-ordinates r', a'; r'', a''; and P any other point whose co-ordinates are r and θ;

$$\text{then } 1 = \frac{PP'' - PP'}{P'P''} = \frac{PP''}{P'P''} - \frac{PP'}{P'P''};$$

$$\text{but } \frac{PP''}{P'P''} = \frac{AP \sin PAP''}{AP' \sin P'AP''} = \frac{r \sin(\theta - \alpha'')}{r' \sin(\alpha' - \alpha'')};$$

$$\text{similarly, } \frac{PP'}{P'P''} = \frac{r \sin(\theta - \alpha')}{r'' \sin(\alpha' - \alpha'')};$$

$$\therefore \frac{r \sin(\theta - \alpha'')}{r' \sin(\alpha' - \alpha'')} - \frac{r \sin(\theta - \alpha')}{r'' \sin(\alpha' - \alpha'')} = 1.$$

10. To find the locus of the vertices of equilateral triangles which have one extremity of their bases in a fixed point, and the other in a given line.

Let $SX = a$ (fig. 25) be a perpendicular from the given point upon the given line; join SM, and let SPM be one of the equilateral triangles; $SP = r$, $\angle XSP = \theta$; then $r = SM = a \sec XSM = a \sec\left(\theta - \frac{\pi}{3}\right)$, the equation required, which represents a straight line.

11. To shew that the equation $x^2y^2 = a(x^3 + y^3)$ will become solvable with respect to y, if the axes of the coordinates be moved through half a right angle.

In that case $\cos\theta = \sin\theta = \frac{1}{\sqrt{2}}$, and

$$x = \frac{1}{\sqrt{2}}(x' - y'),\ y = \frac{1}{\sqrt{2}}(x' + y');\ \text{(Art. 40)}$$

therefore the proposed equation becomes

$$\left(\frac{x'^2 - y'^2}{2}\right)^2 = \frac{a}{2\sqrt{2}}\{(x' - y')^3 + (x' + y')^3\} = \frac{a}{\sqrt{2}}(x'^3 + 3x'y'^2),$$

which can evidently be solved with respect to y'; and it shews that the new axis of x' is an Axis of the curve.

By the same transformation the equation $3axy = x^3 + y^3$

$$\text{becomes } \frac{3a}{\sqrt{2}}(x'^2 - y'^2) = x'^3 + 3x'y'^2;$$

and the new axis of x' is an Axis of the curve.

12. To express by polar co-ordinates the equation

$$y^2 = 4a(a + x) + \tan^2\alpha\,(2a + x)^2.$$

Adding x^2 to both sides we get

$$x^2 + y^2 = (2a + x)^2(1 + \tan^2\alpha);$$

$$\therefore\ r^2\cos^2\alpha = (2a + r\cos\theta)^2,$$

$$\text{or}\quad \pm r\cos\alpha = 2a + r\cos\theta.$$

13. The locus of the middle points of all chords of a circle that subtend a right angle at a fixed point in its plane, is another circle (fig. 91).

Take O the fixed point for the origin and two lines OE, OF, at right angles to one another and cutting the circle, for the axes; and let its equation be

$$(x - a)^2 + (y - b)^2 = c^2;$$

$$\text{then}\quad OE = y' = b \pm \sqrt{c^2 - a^2},$$

$$OF = x' = a \pm \sqrt{c^2 - b^2}.$$

Let G be the middle point of any chord EF, and D the middle point of OC, C being the centre of the circle; then

$$DG^2 = \tfrac{1}{4}(y' - b)^2 + \tfrac{1}{4}(x' - a)^2 = \tfrac{1}{4}(2c^2 - a^2 - b^2);$$

therefore the locus of G is a circle, whose centre is the middle point of OC.

14. Three given circles being traced upon a plane, to shew that any three angles, each of which contains two of the circles, will have their vertices in a straight line.

Let A, B, C, be the centres of the three circles (fig. 92), a, b, c, their radii; and let lines touching the two circles about A and B meet in F, and those touching the circles about B and C meet in D; join FD and produce AC to meet it in E. Then is E the intersection of two lines touching the circles about A and C; for

$$\frac{AE}{CE} = \frac{AE}{BG}\cdot\frac{BG}{CE} = \frac{FA}{FB}\cdot\frac{BD}{CD} = \frac{a}{b}\cdot\frac{b}{c} = \frac{a}{c}.$$

15. To find the co-ordinates of the points of intersection of two circles which cut one another.

Let their equations be

$$x^2 + y^2 = c^2, \quad (x - a)^2 + y^2 = c'^2,$$

so that the axis of x joins the centres; then it may be easily proved that the co-ordinates of their points of intersection are

$$x = \frac{a^2 + c^2 - c'^2}{2a},$$

$$y = \pm \frac{1}{2a} \sqrt{(a + c + c') \cdot (a + c - c') \cdot (a + c' - c) \cdot (c + c' - a)};$$

from which all the common theorems in geometry, relative to the intersections and contacts of circles with one another, may be deduced.

16. In the sides $AX = a$, $AP = b$, (fig. 14) of a given triangle APX, take two points M, N, such that

$$\frac{XM}{MA} = \frac{AN}{NP} = n,$$

and join MN; then all the circles described about AMN, for different values of n, will have a common chord.

It will be found that the equation to the circle referred to AX, AP, as axes, putting $\angle XAP = \omega$, is

$$(x^2 + 2xy \cos\omega + y^2)(n + 1) - ay - nbx = 0;$$

and in order that this may be satisfied by values of x and y independent of n, we must have

$$x^2 + 2xy \cos\omega + y^2 = ay = bx.$$

These equations give real values of x and y, and consequently determine a point in the circle whatever n be; therefore all the circles have a common chord; and its equation is $ay = bx$.

17. If in a regular polygon of n sides, lines be drawn from the extremities of a side to those of any other side so as to cross each other, the locus of their intersection is a circle, whose radius $= \frac{1}{2} a \operatorname{cosec} \frac{2\pi}{n}$.

Let AB, BE, CD, be three sides of the polygon; produce AB to F, and let AD, BC, intersect in P; then since arc AB = arc CD (fig. 93),

$$\angle PAB = CBE; \text{ and } \angle EBF = \frac{2\pi}{n};$$

$$\therefore \angle APB = \frac{2\pi}{n}, \text{ and locus of } P \text{ is a circle};$$

and $AB = a$ is a chord subtending an $\angle \frac{4\pi}{n}$, at its centre;

$$\therefore \text{ its radius } = \frac{1}{2} a \operatorname{cosec} \frac{2\pi}{n}.$$

18. To find the locus of a point from which, if three perpendiculars be dropped on three given straight lines, the points of intersection shall always lie in a straight line.

Let the three given straight lines intersect one another in the points A, B, C (fig. 94), $AC = b$, $AB = c$, $\angle BAC = \omega$, x, y, co-ordinates of P referred to AC, AB, as axes. PM, PN perpendiculars on the axes; join MN cutting BC in Q, and join PQ, then PQ is to be perpendicular to BC. The equations to BC and MN are respectively

$$Y = -\frac{c}{b} X + c = -nX + c \text{ (1) suppose,}$$

$$Y = -\frac{y + x \cos \omega}{x + y \cos \omega} X + y + x \cos \omega;$$

therefore the co-ordinates of their intersection Q are

$$X' = \frac{(y - c + x \cos \omega)(x + y \cos \omega)}{(y + x \cos \omega) - n(x + y \cos \omega)}, \quad Y' = -nX' + c.$$

The equation to PQ is $Y - y = \frac{Y' - y}{X' - x}(X - x)$; and as this is to be perpendicular to (1),

$$\therefore 1 - n\cos\omega + (\cos\omega - n)\frac{Y' - y}{X' - x} = 0 \text{ (Art. 33).}$$

Therefore, substituting for Y' and X' their values, and reducing, we get

$$x^2 + y^2 + 2xy\cos\omega = bx + cy,$$

the equation to the circle circumscribed about the triangle ABC.

The converse of this admits of an easy geometrical proof. Drop the three perpendiculars PM, PN, PQ, upon the three lines, and join MQ, QN. Then $\angle MPN$ = supplement of $MAN = BPC$,

$$\therefore BPM = CPN; \text{ consequently } BQM = CQN,$$

and therefore MQ and QN are in the same straight line.

19. To find the diameter of the circle represented by the equation

$$x^2 + 2xy\cos\omega + y^2 = bx + cy.$$

In order that this equation may represent a circle, the axes must be inclined at an angle $= \omega$; also the origin is a point in the circumference; and making $x = 0$, we get $y = AB = c$, (fig. 94); and if $y = 0$, $x = AC = b$; hence diameter of circle $= \frac{BC}{\sin\omega} = \frac{\sqrt{c^2 + b^2 - 2bc\cos\omega}}{\sin\omega}$.

20. If two parallel planes revolve in their own planes in the same direction about fixed points A, B, with equal velocities; the curve traced on the first by a pencil fixed perpendicularly in the second at a given point P, will be a circle.

Let A be the origin (fig. 95) $AB = c$, $BP = a$, $AP = r$, $B'AP = \theta$; AB' being the position at any time of that fixed

line of the first plane which coincided with AB, at the instant that P was also in AB. Then since PB is parallel to AB',

$$-\cos\theta = \cos APB = \frac{a^2 + r^2 - c^2}{2ar};$$

or $r^2 + 2ar\cos\theta + a^2 = c^2$, the equation to a circle.

If the planes revolve in contrary directions, it may be similarly shewn that the locus of P has for its equation

$$r^2 - 2ar\cos\theta + a^2 = \frac{1}{c^2}(a^2 - r^2)^2.$$

21. To find the equation to the curve traced by a nail projecting out of a vertical wall on a circular board that rolls on level ground at the foot of the wall with its plane vertical.

Let N (fig. 96) be the nail, Ax, Ay, the co-ordinate axes fixed in the board, and suppose the axis of y at the commencement of the motion to be vertical and to pass through N; the board is supposed to have rolled through an $\angle xAQ = \phi$, where AQ is horizontal; $xAN = \theta$, $AN = r$, $NB = a$, $AQ = c$; then $AB = c\phi$ = horizontal space passed over by the centre;

$$\therefore\ r^2 = a^2 + c^2\phi^2;\ \text{ but } \angle NAM = NAB - xAQ;$$

therefore the required equation is

$$\theta = \sin^{-1}\frac{a}{r} - \frac{1}{c}\sqrt{r^2 - a^2}.$$

22. If r, r', be the radii of two circles in one plane and a the distance of their centres, to find the locus of the points from which the two circles appear equally large.

Let a common tangent to the two circles intersect the line joining their centres at a distance d from the centre of the smaller circle whose radius is r, then $d = ar \div (r' - r)$. Let x, y, be co-ordinates of any point of the locus, referred to the above point of intersection as origin, x being parallel and y perpendicular to the line joining the two centres; then

$$\frac{(x - d)^2 + y^2}{(d + a - x)^2 + y^2} = \frac{r^2}{r'^2};$$

$$\therefore\ (x^2+y^2)(r'^2-r^2)-2x\{r'^2d-r^2(d+a)\}=0,$$

$$\text{or } x^2+y^2-2x\,.\,\frac{arr'}{r'^2-r^2}=0,$$

the equation to a circle.

23. Having given the lengths of two tangents to a parabola at right angles to one another, to find its latus rectum.

Let $QP=b$, $QP'=c$, (fig. 31)

$$\text{then } \frac{2a}{SP}=1+\cos\theta,\quad \frac{2a}{SP'}=1-\cos\theta;$$

therefore by Arts. 83 and 86, $a\,.\,\dfrac{PP'}{SQ^2}=1$,

$$\text{or } 4a=4\frac{SQ^2\,.\,P'P^2}{P'P^3}=\frac{4b^2c^2}{(b^2+c^2)^{\frac{3}{2}}}.$$

24. To find the locus of the intersection of two normals to a parabola at right angles to one another.

If m be the tangent of the inclination of the normal to the axis of x, its equation is (Art. 80)

$$y+2am=m(x-am^2);$$

and changing m into $-\dfrac{1}{m}$, the equation to a second normal perpendicular to it is

$$y-\frac{2a}{m}=-\frac{1}{m}\left(x-\frac{a}{m^2}\right).$$

Hence, adding and subtracting, we get

$$y=a\left(\frac{1}{m}-m\right),\quad x-3a=a\left(\frac{1}{m}-m\right)^2;$$

$$\therefore\ y^2=a(x-3a),$$

the equation to the required locus, which is an equal parabola.

25. To draw normals to a parabola through a given point.

Let h, k, be the co-ordinates of the given point, x, y those of a point in the curve through which one of the normals passes; then

$$k - y = -\frac{y}{2a}(h - x), \quad y^2 = 4ax;$$

and eliminating x, we get $y^3 - 4a(h - 2a)y - 8a^2k = 0$, which compared with $y^3 + qy + r = 0$, gives

$$\frac{r^2}{4} + \frac{q^3}{27} = 16a^4\left\{k^2 - \frac{4}{27a}(h - 2a)^3\right\};$$

hence (Theory of Equations, Art. 90), when $k^2 = \frac{4}{27a}(h-2a)^3$, two normals may be drawn through (h, k), for then y has only two real different values; and according as $k^2 >$ or $< \frac{4}{27a}(h - 2a)^3$, y has only one, or three real distinct values; and one or three normals can be drawn through the given point. This is the same thing as saying that through a given point one, two, or three normals may be drawn according as the point lies without, upon, or within the evolute.

26. To find the radius of the circle which passes through the intersections of the tangents at the extremities of three given focal distances of a parabola.

The diameter of the circle (since it is described about the triangle MLS) $= \frac{MS}{\sin MLS} = \frac{SM.SL}{SD}$

$$= \frac{\sqrt{SP.SP'}.\sqrt{SP.SQ}}{\sqrt{SP.SA}} = \sqrt{\frac{SP.SQ.SP'}{SA}}$$

(fig. 77).

27. If one side of a triangle and two others produced be tangents to a parabola, and the points of contact be joined, a

triangle will be formed whose area is double of that of the exterior triangle.

$$\text{Segment } PQR = \tfrac{2}{3}\Delta PTR \text{ (fig. 97) (Art. 103)},$$
$$\text{or } \Delta PQR + \tfrac{2}{3}(\Delta PUQ + \Delta QVR) = \tfrac{2}{3}\Delta PTR;$$
$$\therefore\ \Delta PQR = \tfrac{2}{3}(\Delta TUV + \Delta PQR),$$
$$\text{or } \Delta PQR = 2\Delta TUV.$$

28. If in any segment of a parabola a polygon be inscribed having the same base as the segment, the sum of the cube roots of the areas of all the partial segments standing upon the sides of the polygon, is equal to the cube root of the area of the whole segment.

Let y, y', y'', be the ordinates of three points P, Q, R, (fig. 98); take s the middle point of the chord PQ, and draw msn parallel, and Qn, Mm, Ns perpendicular to the axis Ax;

$$\text{then } sN = \tfrac{1}{2}(y + y'),\ \therefore\ AM = \frac{(y+y')^2}{16a};$$

$$\text{and } AN = \frac{y^2 + y'^2}{8a},$$

$$\therefore\ MN = ms = \frac{(y'-y)^2}{16a};$$

$$\text{also } Qn = \tfrac{1}{2}(y' - y),$$

$$\therefore \text{ segment } PQ = \tfrac{4}{3} ms \,.\, Qn = \frac{(y'-y)^3}{24a};$$

similarly,

$$\text{segment } QR = \frac{(y''-y')^3}{24a},\ \text{segment } PR = \frac{(y''-y)^3}{24a};$$

$$\therefore\ \sqrt[3]{\text{seg. } PQ} + \sqrt[3]{\text{seg. } QR} = \sqrt[3]{\text{seg. } PR}.$$

29. Two tangents a, b, to a parabola intersect at an angle $= \omega$, and a circle is described between the tangents and the curve; to find its diameter.

The equations to the parabola, and circle, referred to the tangents as axes, are

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1, \quad x + y - 2\sqrt{xy}\sin\tfrac{1}{2}\omega = r\cot\tfrac{1}{2}\omega;$$

therefore for the points of intersection of the curves,

$$x + b\left(1 - \sqrt{\frac{x}{a}}\right)^2 - 2\sin\tfrac{1}{2}\omega\sqrt{xb}\left(1 - \sqrt{\frac{x}{a}}\right) = r\cot\tfrac{1}{2}\omega;$$

and in order that the circle may touch the parabola, the two values of x must be equal in this equation,

$$\therefore (a + b + 2\sqrt{ab}\sin\tfrac{1}{2}\omega)(b - r\cot\tfrac{1}{2}\omega)a$$
$$= ab(b + 2\sqrt{ab}\sin\tfrac{1}{2}\omega + a\sin^2\tfrac{1}{2}\omega),$$

which gives $2r = \dfrac{ab\sin\omega}{a + b + 2\sqrt{ab}\sin\frac{1}{2}\omega}$.

30. If one side of a triangle and the other two produced be tangents to a parabola, and each of the angles of the triangle be joined with the point in which the parabola touches the opposite side; the three straight lines thus drawn will intersect one another in a point the locus of which, for different parabolas, is an ellipse circumscribing the triangle.

ABC the given triangle (fig. 99), *RPS* a parabola touching the base in *P* and the two sides produced in *R*, *S*; $AR = r$, $AS = s$, and a, b, c the sides of the triangle. Then taking *AR*, *AS* as axes, the equations to the parabola and to its tangent at *P* (h, k), are

$$\sqrt{\frac{x}{r}} + \sqrt{\frac{y}{s}} = 1, \quad \frac{x}{\sqrt{rh}} + \frac{y}{\sqrt{sk}} = 1;$$

$$\therefore c = \sqrt{sk}, \quad b = \sqrt{rh}; \text{ and } \frac{b}{r} + \frac{c}{s} = 1 \quad (1).$$

Now the equations to *BR*, *CS*, are respectively

$$\frac{x}{r} = 1 - \frac{y}{c}, \quad \frac{y}{s} = 1 - \frac{x}{b} \quad (2);$$

therefore for their point of intersection O we get

$$\frac{y}{s} - \frac{x}{r} = \frac{y}{c} - \frac{x}{b};$$

$$\therefore \frac{y}{x}\left(\frac{1}{c} - \frac{1}{s}\right) = \frac{1}{b} - \frac{1}{r} \text{ or } \frac{y}{x}\cdot\frac{b}{cr} = \frac{c}{sb}; \quad \therefore \frac{y}{x} = \frac{c^2 r}{b^2 s} = \frac{k}{h},$$

which shews that the line joining A and O passes through P. Also eliminating r and s between (1) and (2),

$$\text{since } xy\left(\frac{b}{r} + \frac{c}{s}\right) = xy,$$

$$\text{we get } by\left(1 - \frac{y}{c}\right) + cx\left(1 - \frac{x}{b}\right) = xy,$$

the equation to the locus of O, which represents an ellipse passing through the points A, B, C.

31. To describe a parabola which shall touch three straight lines, one of them in a given point.

Let R be the given point in AC (fig. 99), and AB, BC, the two other given lines touched by the parabola in the points S and P.

Then using the notation of the preceding problem, we shall obtain the equation to the parabola referred to PB and Px which is parallel to AR, as axes, by putting

$$x = x' - \frac{b}{a}y' + \frac{b^2}{r}, \quad y = \frac{c}{a}y' + \frac{c^2}{s};$$

therefore the equation is, suppressing the accents,

$$\frac{cy}{as} + \frac{c^2}{s^2} = \left\{1 - \left(\frac{x}{r} - \frac{by}{ar} + \frac{b^2}{r^2}\right)^{\frac{1}{2}}\right\}^2;$$

or, reducing, $y = \dfrac{a}{r}x \pm \dfrac{2a}{r}\sqrt{(r-b)x}$.

Hence, proceeding as in Ex. 5, Art. 234, if PV be the diameter whose equation is $y = \dfrac{a}{r}x$, and $\angle\, BPV = \alpha$, the latus rectum

$$= \frac{4a^2}{r^2}(r-b)\frac{\sin^3 a}{\sin C} = \frac{4a^2 r(r-b)\sin^2 C}{(r^2 - 2ar\cos C + a^2)^{\frac{3}{2}}} \quad (1).$$

$$\text{or } L = 4\sin^2 A . \sqrt{bcrs}\frac{(r-b)(s-c)}{(rb + sc - a^2)^{\frac{3}{2}}}.$$

32. To describe a parabola touching three given straight lines so that its latus rectum may be the greatest possible.

Since the latus rectum vanishes when $r = b$, and also when $s = c$, it must admit of a maximum as the point P moves from C to B (for the focus describes a circle passing through A, B, C), and the corresponding value of r is given by the equation

$$r^3 + r^2(a\cos C - 2b) + ar(b\cos C - 2a) + ba^2 = 0;$$

which will have three real roots, as each side will be touched by a parabola whose latus rectum is a maximum.

Hence if the magnitude of the latus rectum be given, each side will be touched by two parabolas, having latera recta of that magnitude; and we see that equation (1) for finding r would be of the sixth degree.

If we suppose the two lines AB, BC to become coincident, then S and P coincide with B, and the parabola touches BC in a given point for which $BC = a$,

$$\text{and latus rectum} = \frac{4a^2 r^3 \sin^2 C}{(r^2 - 2ar\cos C + a^2)^{\frac{3}{2}}}.$$

33. The directrix of every parabola that touches three given straight lines passes through the intersection of the perpendiculars dropped from the intersection of every two lines upon the remaining one (fig. 94).

P a point in the circumference of a circle circumscribing triangle ABC, and therefore the focus of a parabola touching the sides AB, AC, BC; x, y, its co-ordinates referred to AC, AB as axes; PM, PN perpendicular to AB, AC; then the equation to MN, which is a tangent at vertex of parabola, is

$$Y = mX + c, \text{ when } m = -\frac{y + x\cos\alpha}{x + y\cos\alpha};$$

$\therefore$ equation to a line through P parallel to MN is $Y - y = m(X - x)$.

Let $M'N'$ be the directrix, and therefore parallel to MN and at the same distance from it as P is;

$$\therefore\ AN' = 2(x + y\cos\alpha) - x + \frac{y}{m} = x + \left(2\cos\alpha + \frac{1}{m}\right)y,$$

$$AM' = 2(y + x\cos\alpha) - y + mx = y + (2\cos\alpha + m)x,$$

or, substituting for m its value,

$$AN' = \frac{(x^2 + y^2 + 2xy\cos\alpha)\cos\alpha}{y + x\cos\alpha},$$

$$AM' = \frac{(x^2 + y^2 + 2xy\cos\alpha)\cos\alpha}{x + y\cos\alpha};$$

$$\therefore\ X(y + x\cos\alpha) + Y(x + y\cos\alpha)$$

$$= (bx + cy)\cos\alpha \text{ is the equation to } N'M',$$

($\because$ the equation to the circle is $x^2 + y^2 + 2xy\cos\alpha = bx + cy$)

which equation is satisfied, whatever be x and y, by

$$X + Y\cos\alpha = c\cos\alpha,$$

$$X\cos\alpha + Y = b\cos\alpha;$$

$\therefore \left.\begin{aligned} X\sin^2\alpha &= \cos\alpha\,(c - b\cos\alpha) \\ Y\sin^2\alpha &= \cos\alpha\,(b - c\cos\alpha) \end{aligned}\right\}$ which determine the intersection of the perpendiculars.

34. If the vertex and nearer focus of an ellipse be fixed, whilst the centre assumes all positions in the indefinite line passing through them, the curve will successively become a parabola, circle, limited straight line, hyperbola, unlimited straight line, hyperbola, and parabola.

Calling $AS = p$, $SC = c$, and taking the vertex for origin, the equation is

$$y^2 = \frac{p+2c}{p+c}\left(2px - \frac{px^2}{p+c}\right);$$

and it assumes the following forms for different values of c;

$c = \infty$, $y^2 = 4px$ the limiting parabola,

$c = 0$, a circle.

When c is negative, the equation is

$$y^2 = \frac{p-2c}{p-c}\left(2px - \frac{px^2}{p-c}\right); \text{ and for}$$

$c < \frac{1}{2}p$, the curve is an ellipse,

$c = \frac{1}{2}p$, a straight line coincident with axis of x, being the limit of the ellipse,

$c > \frac{1}{2}p < p$, a hyperbola cutting the limiting parabola,

$c = p$, an unlimited line touching that parabola at its vertex,

$c > p$, a hyperbola exterior to the limiting parabola; and for

$c = \infty$, the curve is again the limiting parabola.

35. To find the locus of one end of a given straight line, whose other end, and a given point in it, move in straight lines at right angles to one another.

AP the given line $= a$, B the given point in it, $PB = b$, $CN = x$, $NP = y$, $\angle PBN = \theta$; (fig. 101) then $x = a\cos\theta$, $y = b\sin\theta$,

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ the equation to an ellipse.}$$

If the rectangle CO be completed, and PO joined, PO is a normal to the ellipse at P; for

$$\frac{GN}{CN} = \frac{GN}{BN}\cdot\frac{BN}{CN} = \frac{b}{a}\cdot\frac{b}{a} = \frac{b^2}{a^2}.$$

Hence if a line, whose length is the semi-major axis of an ellipse, have one end in the curve, and the other in the minor axis; then (1) the part cut off by the major axis will always equal the semi-axis minor, and (2) the locus of the intersection of the perpendicular to the minor axis through one end, with the normal through the other, will be a circle.

36. Two given circles touch each other internally; to shew that the locus of the centre of the circle which touches each of them, is an ellipse having their centres for its foci.

S and H the centres of the circles, P the centre of a circle touching both; join SP passing through the point of contact B, and HP passing through the point of contact A (fig. 102);

$$\text{then } SP + HP = SB + BP + (HA - AP) = SB + HA,$$

which is constant; therefore the locus of P is an ellipse.

37. If α, β, γ be the angles which the transverse axis, and the focal and central distances of any point of a curve of the second order, make with the tangent at that point,

$$\tan\alpha \,.\, \tan\gamma = \tan^2\beta.$$

Let $SPY = \beta$, $CPY = \gamma$, $STP = \alpha$, (fig. 41)

$$\text{then } e = \frac{SG}{SP} = \frac{\cos\beta}{\cos\alpha};$$

$$\text{and } \frac{NG}{CN} = \tan\alpha \,.\, \tan(\gamma - \alpha) = 1 - e^2 = 1 - \frac{\cos^2\beta}{\cos^2\alpha};$$

therefore, reducing, we get $\tan\alpha\tan\gamma = \tan^2\beta$.

38. The products of the alternate segments of the sides of a polygon described about an ellipse are equal.

Let p, q, r, s be the lengths of the semi-diameters respectively parallel to the four tangents at P, Q, R, S, (fig. 103) the proof being the same whatever the number of sides;

$$\text{then } \frac{O_1P}{O_1Q} = \frac{p}{q}, \quad \frac{O_2R}{O_2P} = \frac{r}{p}, \quad \frac{O_3S}{O_3R} = \frac{s}{r}, \quad \frac{O_4Q}{O_4S} = \frac{q}{s}, \quad \text{(Art. 155)};$$

$$\therefore\; O_1P \,.\, O_2R \,.\, O_3S \,.\, O_4Q = O_1Q \,.\, O_2P \,.\, O_3R \,.\, O_4S.$$

In the case of a triangle ABC whose sides touch the ellipse in the points α, β, γ, we should get $A\beta . C\alpha . B\gamma = A\gamma . B\alpha . C\beta$, which shews that the three lines joining the points of contact with the opposite angles, intersect in a point.

39. If θ be the acute angle between the tangent and focal distance at any point of an ellipse, the distance of that point from the centre is $\sqrt{a^2 - b^2 \cot^2\theta}$ (fig. 42).

$$\text{For } r^2 = a^2 - b^2\left(\frac{a^2}{p^2} - 1\right) = a^2 - b^2\left(\frac{PE^2}{CQ^2} - 1\right) = a^2 - b^2 \cot^2\theta.$$

40. To find the locus of the intersection of the normal to an ellipse and the perpendicular upon it from the centre. The equation to the normal at any point is

$$(y - mx)\sqrt{a^2 + m^2 b^2} + (a^2 - b^2)\, m = 0,$$

and the equation to a perpendicular upon it from the centre is

$$y = -\frac{1}{m}x, \text{ which gives } m = -\frac{x}{y};$$

therefore, substituting, the equation to the required locus is

$$(x^2 + y^2)\sqrt{a^2 y^2 + b^2 x^2} = (a^2 - b^2)\, xy.$$

41. A given triangle has always two of its angular points in two given straight lines; to find the locus of the remaining angular point.

Take the given lines for the axes, and let

$$\angle AOB = \omega,\ CM = x,\ CN = y; \quad \angle OAB = \phi,\ \text{(fig. 104)};$$

$$\therefore\ y \sin\omega = b \sin(A + \phi),$$

$$x \sin\omega = a \sin(B + \pi - \phi - \omega) = -a \sin(A + \phi + C + \omega),$$

$$\therefore\ bx \sin\omega = -a\cos(C + \omega) . y \sin\omega - ab\cos(A + \phi)\sin(C + \omega),$$

$$\therefore\ \sin^2\omega\,\{bx + a\cos(C + \omega)\, y\}^2 = a^2 \sin^2(C + \omega)\,(b^2 - y^2 \sin^2\omega),$$

the equation to an ellipse of which O is the centre.

42. To find the locus of the middle point of a chord of constant length in an ellipse.

Let $QV = c$, $CV = r$, $ACV = \theta$, (fig. 45); then

$$\frac{QV^2}{CD^2} + \frac{CV^2}{CP^2} = 1.$$

But, $CP^2 = \dfrac{b^2}{1 - e^2 \cos^2\theta}$,

$$CD^2 = a^2 + b^2 - CP^2 = \frac{a^2 - (a^2 + b^2)\, e^2 \cos^2\theta}{1 - e^2 \cos^2\theta},$$

$$\therefore \frac{c^2 (1 - e^2 \cos^2\theta)}{a^2 - (a^2 + b^2)\, e^2 \cos^2\theta} + \frac{r^2 (1 - e^2 \cos^2\theta)}{b^2} = 1,$$

the polar equation to the required locus.

43. Two given circles are traced upon a plane and a line is drawn touching one and cutting the other in two points at which tangents are applied to the latter circle; to find the locus of the intersection of the tangents.

$OP = r$, $\angle POO' = \theta$, $OO' = c$, $OQ = a$, $O'Q' = a'$ (fig. 69),

$$\text{then } c \cos\theta - a' = ON = \frac{OQ^2}{OP} = \frac{a^2}{r};$$

$$\therefore r = \frac{a^2}{c\cos\theta - a'},$$

the equation to a conic section of which O is a focus.

44. Having given the base and altitude of a triangle, to find the locus of the centre of the inscribed circle.

$SC = CH = c =$ half given base (fig. 41), $PN = a$ the given altitude, O the centre of the inscribed circle, $CM = x$, $MO = y$ its co-ordinates, $CN = x'$; then

$$\tan S = \frac{a}{c + x'},\ \tan\frac{S}{2} = \frac{y}{c + x},\ \tan H = \frac{a}{c - x'},\ \tan\frac{H}{2} = \frac{y}{c - x};$$

$$\therefore \frac{a}{c + x'} = \frac{2y(c + x)}{(c + x)^2 - y^2},\ \frac{a}{c - x'} = \frac{2y(c - x)}{(c - x)^2 - y^2};$$

therefore inverting and adding in order to eliminate x', we get the required equation, which is of the third degree,

$$\frac{2}{a} = \frac{1}{y} - \frac{y}{c^2 - x^2}.$$

45. Having given the base of a triangle, and the sum of the other two sides, to find the locus of the centre of the inscribed circle.

SH the given base $= 2ae$, $SP + PH = 2a$, and $SO = r$, $HSO = \theta$, (fig. 41), polar co-ordinates of the describing point O; then area of triangle $= \frac{1}{2}$ perimeter $\times$ radius of inscribed circle

$$= a(1 + e)\, r \sin\theta,$$

also area of triangle $= SP \sin 2\theta \,.\, ae = \dfrac{a^2 e(1 - e^2)\sin 2\theta}{1 - e\cos 2\theta}$;

$$\therefore\ r = \frac{2ae(1 - e)\cos\theta}{1 - e\cos 2\theta},$$

the equation to a conic section of which SH is the major axis.

46. Two focal distances of a conic section include a constant angle β, and one of them is produced to meet the tangent at the extremity of the other, to find the locus of the point of intersection.

$$ST = r,\ \angle AST = \theta,\ PSQ = \beta;\ \text{(fig. 30), then}$$

$$ASP = \beta + \theta, \text{ and } \therefore\ r = \frac{a(1 - e^2)}{\cos\beta + e\cos\theta}\ \text{(Art. 127)},$$

the equation to a conic section with focus S; and which is an ellipse, hyperbola, or parabola, according as $\cos\beta >, <,$ or $= e$.

If we draw another focal distance SP' inclined at angle β to SQ, then the tangent at P' will pass through T (Art. 128); therefore the preceding is also the solution of the problem to find the locus of the intersection of tangents to an ellipse at the extremities of two focal distances that include a constant angle 2β.

Also, if TP be produced to T' so that $\angle PST' = PST$, then T' is a point in the locus of T; from whence it follows that the chords of a conic section whose eccentricity $= e$, that subtend an angle 2β at the focus, will be tangents to another conic section having the same focus whose eccentricity $= e\cos\beta$.

47. In an ellipse, if two focal distances r, r', include an angle $= 2\beta$, and T be the intersection of tangents at their extremities, then

$$ST^2 = \frac{b^2 rr'}{b^2 - rr' \sin^2\beta}.$$

We have $\dfrac{a(1-e^2)}{r} = 1 + e\cos\theta$, $\dfrac{a(1-e^2)}{r'} = 1 + e\cos\theta'$,

and $\dfrac{a(1-e^2)}{ST} = \cos\beta + e\cos AST = \cos\frac{1}{2}(\theta'-\theta) + e\cos\frac{1}{2}(\theta+\theta')$;

between which equations if we eliminate θ and θ', we shall obtain the above result; which in the case of the parabola becomes $ST^2 = rr'$ agreeably to Art. 83.

48. To find the equation to the curve traced out by a point in the perimeter of a circle which rolls upon another equal circle.

Let A' be the describing point, at first in contact with A, and AA' the curve traced out, (fig. 70); C, C' the centres of the circles; join AA', CC', and let

$$AA' = r, \quad A'AE = \theta, \quad AC = a.$$

AA' is manifestly parallel to CC', draw DA' parallel to AC, and therefore $= AC$;

$$\text{then } AA' = CD = CC' - DC',$$

$$\text{or } r = 2a - 2a\cos\theta, \text{ the polar equation.}$$

Or if $AR = x$, $RA' = y$, be the rectangular co-ordinates of A',

$$\sqrt{x^2+y^2} = 2a\left(1 - \frac{x}{\sqrt{x^2+y^2}}\right);$$

$$\therefore x^2 + y^2 = 2a(\sqrt{x^2+y^2} - x).$$

49. To find the equation to the curve described by a point in the perimeter of a circle which rolls within another circle of four times its radius.

P the describing point, at first in contact with A (fig. 70), and AP the curve traced out;

$$CN = x, \quad PN = y, \quad CA = 4a, \quad QO = a,$$

O being the centre of the rolling circle,

$ACD = \theta$; therefore $QOP = 4\theta$, and consequently, OM being perpendicular to AC,

$$\angle POM = \pi - \left(\frac{\pi}{2} - \theta\right) - 4\theta = \frac{\pi}{2} - 3\theta;$$

$$\therefore\ x = CM + Pn = 3a\cos\theta + a\cos 3\theta = 4a(\cos\theta)^3,$$

$$y = OM - On = 3a\sin\theta - a\sin 3\theta = 4a(\sin\theta)^3;$$

$$\therefore\ \left(\frac{x}{4a}\right)^{\frac{2}{3}} + \left(\frac{y}{4a}\right)^{\frac{2}{3}} = 1.$$

Suppose $CA = 2a$, the radius of the rolling circle equal a, and the point P to be not in the circumference of the rolling circle, but at a distance c from its centre; then

$$\angle POM = \tfrac{1}{2}\pi - \theta, \quad \therefore\ x = (a + c)\cos\theta, \quad y = (a - c)\sin\theta;$$

$$\therefore\ \frac{x^2}{(a+c)^2} + \frac{y^2}{(a-c)^2} = 1,$$

the equation to an ellipse; except $a = c$ when the equation is $y = 0$, and the locus is the axis of x.

50. If a triangle be inscribed in a Conic Section, and each side be produced to meet the tangent at the opposite angle, the three points of intersection will lie in a straight line.

If we take C for the origin, and $AC = b$, $BC = a$, for axes, the equation to the conic section must be of the form $x^2 + my^2 + nxy - ax - mby = 0$.

The equation to the tangent at A is

$$y - b = -\frac{nb - a}{mb}x; \text{ and when } y = 0,\ x = \frac{mb^2}{nb - a},$$

which determines one of the points. The equation to the tangent at B is $y = -\dfrac{a}{na - mb}(x - a)$, and when $x = 0$, $y = \dfrac{a^2}{na - mb}$ which determine a second point. The equation to the tangent at C is $y = \dfrac{-ax}{mb}$, and equation to AB is $\dfrac{x}{a} + \dfrac{y}{b} = 1$, and for their intersection $x = \dfrac{mab^2}{mb^2 - a^2}$, $y = -\dfrac{a^2 b}{mb^2 - a^2}$ the co-ordinates of the third point. Now the equation to the line joining the first and second point is

$$\frac{na - mb}{a^2}y + \frac{nb - a}{mb^2}x = 1,$$

and this is evidently satisfied by the co-ordinates of the third point.

51. An ellipse being referred to conjugate diameters, if with the co-ordinates of any point as conjugate semi-diameters a second ellipse be described, it will be touched by the chord of the former that joins the extremities of the diameters.

The equation to the interior ellipse will be

$$\frac{x^2}{h^2} + \frac{y^2}{k^2} = 1, \text{ with condition } \frac{h^2}{a^2} + \frac{k^2}{b^2} = 1;$$

and for the intersection of this ellipse with the chord

$\dfrac{x}{a} + \dfrac{y}{b} = 1$, we have $\dfrac{x^2}{h^2} + \dfrac{b^2}{k^2}\left(1 - \dfrac{x}{a}\right)^2 = 1$, or $(ax - h^2)^2 = 0$,

which shews that the chord is a tangent at a point for which $x = \dfrac{h^2}{a}$.

52. The chords joining the extremities of conjugate diameters of an ellipse will all touch in their middle points a similarly situated ellipse with axes $a\sqrt{2}$, $b\sqrt{2}$.

The equation to the inner ellipse and to the chord, referred to a pair of conjugate diameters of the outer ellipse as axes, will be respectively,

$$2\left(\frac{x^2}{a'^2}+\frac{y^2}{b'^2}\right)=1, \quad \frac{x}{a'}+\frac{y}{b'}=1\,;$$

therefore for their intersection we have,

$$2\left(\frac{x^2}{a'^2}+1-2\frac{x}{a'}+\frac{x^2}{a'^2}\right)=1, \text{ or } \left(\frac{2x}{a'}-1\right)^2=0\,;$$

which shews that the chord is a tangent in its middle point.

53. To find the locus of the intersection of two tangents to an ellipse applied at the extremities of a chord which always touches a concentric and similarly situated ellipse.

Let a, b, be the semi-axes of the exterior ellipse, a', b', those of the interior; and (h, k) the point through which two tangents to the former pass; then the equation to the chord joining the points of contact is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1,$$

which must be identical with $\dfrac{x'x}{a'^2}+\dfrac{y'y}{b'^2}=1$,

the equation to a line touching the interior ellipse at a point (x', y'); therefore

$$\frac{h}{a^2}=\frac{x'}{a'^2};\quad \frac{k}{b^2}=\frac{y'}{b'^2};$$

hence since $\left(\dfrac{x'}{a'}\right)^2+\left(\dfrac{y'}{b'}\right)^2=1$, the equation to the required locus is $\left(\dfrac{a'h}{a^2}\right)^2+\left(\dfrac{b'k}{b^2}\right)^2=1$, representing a similarly situated ellipse with semi-axes $\dfrac{a^2}{a'}$, $\dfrac{b^2}{b'}$.

54. An ellipse whose centre C is given touches a fixed straight line PQY in a given point P; to find the locus of either focus S.

Let $PQ = h$, $CQ = k$ (fig. 105) be the co-ordinates of C, which are known, and $PY = x$, $SY = y$ those of S; then since CY is parallel to PH, $\angle CYQ = SPY$,

$$\therefore \frac{k}{x-h} = \frac{y}{x} \text{ or } \frac{k}{y} + \frac{h}{x} = 1,$$

the equation to a hyperbola.

55. If an ellipse and hyperbola have the same foci, the locus of the intersection of tangents to them, at right angles to one another, is a circle.

The equation to a line touching the ellipse is

$$y - mx = \sqrt{b^2 + m^2 a^2},$$

and changing m into $-\dfrac{1}{m}$, and a^2, b^2, into a'^2, $-b'^2$, the equation to a line touching the hyperbola, and at right angles to the above, is

$$my + x = \sqrt{-m^2 b'^2 + a'^2};$$

where, since the curves have the same foci and centre,

$$SC^2 = a^2 - b^2 = a'^2 + b'^2.$$

Adding the squares of these two equations,

$$(x^2 + y^2)(1 + m^2) = b^2 + a'^2 + m^2(a^2 - b'^2) = (b^2 + a'^2)(1 + m^2),$$

$$\text{or } x^2 + y^2 = b^2 + a'^2,$$

the equation to a circle passing through the four points of intersection of the curves.

56. To find the locus of the centres of the ellipses inscribed in a given quadrilateral.

Take lines through one of the angles of the quadrilateral parallel to two sides for the axes of the co-ordinates; and let $x = h$, $y = k$, $y = mx$, $y = nx$, be the equations to the four sides; then the conditions for these lines, respectively, being tangents to the ellipse, supposing its equation to be

$$ay^2 + bxy + cx^2 + dy + ex + 1 = 0,$$

(found by making the two points coincide in which each cuts the ellipse) are

$$4a(ch^2 + eh + 1) = (bh + d)^2,$$

$$4c(ak^2 + dk + 1) = (bk + e)^2,$$

$$4(am^2 + bm + c) = (dm + e)^2,$$

$$4(an^2 + bn + c) = (dn + e)^2.$$

But if x', y', be the co-ordinates of the centre of the ellipse, the two former become (Art. 223)

$$(4ac - b^2)(h^2 - 2hx') + 4a - d^2 = 0,$$

$$(4ac - b^2)(k^2 - 2ky') + 4c - e^2 = 0;$$

and eliminating b between the two latter, we get

$$(4a - d^2)\,mn = 4c - e^2,$$

$$\therefore\ k^2 - 2ky' = mn\,(h^2 - 2hx')$$

the required equation; which represents the line joining the middle points of the diagonals of the quadrilateral.

That the locus of the centres would pass through the middle points of the three diagonals might have been foreseen; because each of the diagonals may be regarded as the transverse axis of an evanescent ellipse touching the four sides of the quadrilateral.

If one of the angles become equal to two right angles, the ellipses are inscribed in a triangle, touching its base in a given point; and their centres lie in the line joining the middle point of the base, with the middle point of the line drawn from the vertical angle to the common point of contact.

57. In a given triangle to inscribe an ellipse of given area, and touching one of the sides in a given point.

Let P be the given point in the side BC (fig. 106) and M the middle point of BC; draw MS bisecting AP, and AQ cutting off $QC = BP$; then the centres of all the ellipses that

can be inscribed in the triangle and touch BC in P, lie in MS. Let O be the centre of one of these ellipses, and PD a diameter, then RT the tangent at D is parallel to BC, and D falls in AQ; for by Prob. 60 $\sqrt{RD.BP} = \sqrt{DT.PC} = \frac{1}{2}$ diameter conjugate to OP, therefore $\frac{RD}{DT} = \frac{PC}{PB} = \frac{BQ}{QC}$; consequently the loci of D and O are as asserted. Let $DQ = x$, $AQ = k$, $MC = a$, $BP = c$; then

$$\frac{DT}{QC} = \frac{k-x}{k} \text{ or } DT = \frac{c}{k}(k-x),$$

and $OP . \sin OPM = \frac{1}{2} DQ . \sin DQP = \frac{1}{2} x \sin \omega$, suppose.

$$\text{Now } (\text{area})^2 = \pi^2 \sin^2 OPM . PC . DT . OP^2$$

$$= \frac{\pi^2 \sin^2 \omega \, (2ac - c^2)}{4k} x^2 (k-x).$$

If the area of the ellipse equals the area of a circle radius r,

$$\text{then } (2ac - c^2) \sin^2 \omega x^2 (k - x) = 4kr^4$$

is the equation for finding x. For the greatest inscribed ellipse that touches BC in P we must evidently have

$$x = \tfrac{2}{3}k, \text{ or } MO = 2SO.$$

$$\text{Then } r^4 = \frac{1}{27} k^2 \sin^2 \omega \, (2ac - c^2),$$

which is a maximum by the variation of c when $c = a$, or P coincides with M. Hence the greatest of all the inscribed ellipses touches each side in its middle point, and has its centre coincident with the centre of gravity of the triangle.

58. In the equation $ay^2 + bxy + cx^2 + dy + ex + 1 = 0$, suppose b to assume different values, all the other coefficients remaining unchanged; then (1) the conic sections which it represents are in general all described about the same quadrilateral; (2) the locus of their centres is another conic section, whose equation is $2ay^2 + dy = 2cx^2 + ex$; and (3) the centre

of this last conic section is in the middle point of the line joining the bisections of the diagonals of the quadrilateral.

It is evident that the four points in which the curve cuts the co-ordinate axes are independent of b. If h, k, be the co-ordinates of its centre, then

$$2ak + bh + d = 0, \quad 2ch + bk + e = 0,$$

between which eliminating b, we find the locus of the centre to be the conic section, whose equation is

$$2ak^2 + kd = 2ch^2 + he.$$

If h' and k' be the co-ordinates of the centre of this curve,

$$h' = -\frac{e}{4c} = \tfrac{1}{4}(x_1 + x_2), \quad k' = -\frac{d}{4a} = \tfrac{1}{4}(y_1 + y_2), \text{ (Art. 240)}$$

which are the co-ordinates of the middle point of the line joining the bisections of the diagonals. It may be shewn that the curve passes through the intersection of the diagonals, and also through the points of intersection of each pair of opposite sides.

59. To find the locus of the point which is the intersection of three normals to an ellipse.

The equation to the normal at a point (x', y') of an ellipse is

$$y - y' = \frac{a^2}{b^2}\frac{y'}{x'}(x - x'),$$

$$\text{or } yx'\sqrt{1 - e^2} = (x - e^2x')\sqrt{a^2 - x'^2}.$$

Let h, k be co-ordinates of a given point through which the normal passes, then

$$k^2x'^2(1 - e^2) = (h - e^2x')^2(a^2 - x'^2) \dots\dots\dots (1)$$

is the equation for determining x', the abscissa of the point in the ellipse; and as this equation is of the form

$$e^4x'^4 - \&c. - a^2h^2 = 0,$$

it has two or four possible roots; and consequently through the point (h, k) in general either four or two normals can be drawn. If two of the possible roots become equal (which can only happen in the case of four real roots), then three normals will pass through the point (h, k); in that case the derived equation

$$k^2 x'(1-e^2) = -e^2(h-e^2x')(a^2-x'^2) - x'(h-e^2x')^2$$

has one of them. Dividing this by (1), we find

$$\frac{1}{x'} = -\frac{e^2}{h-e^2x'} - \frac{x'}{a^2-x'^2}, \text{ or } ha^2 = e^2x'^3;$$

$\therefore\ x' = \left(\frac{ha^2}{e^2}\right)^{\frac{1}{3}}$ satisfies equation (1), and substituting we get

$$k^{\frac{2}{3}}(1-e^2)^{\frac{1}{3}} + h^{\frac{2}{3}} = (ae^2)^{\frac{2}{3}}$$

for the equation of condition that (1) may have equal roots, and as often as h and k satisfy this equation, three normals to the ellipse will pass through the point (h, k). The above is consequently the equation to the locus of the intersection of three normals to an ellipse; and coincides, as might have been foreseen, with the equation to the evolute.

60. If the tangents at the extremities of any diameter of an ellipse DD', be intersected by the tangent at any other point, in T, T'; then $DT.D'T' = CP^2$.

The equation to the tangent at Q (fig. 47) is

$$\frac{xx'}{a'^2} + \frac{yy'}{b'^2} = 1;$$

and making $y = b'$ and $-b'$, successively, we get

$$DT.\frac{x'}{a'^2} = 1 - \frac{y'}{b'},\quad D'T'.\frac{x'}{a'^2} = 1 + \frac{y'}{b'};$$

$$\therefore\ DT.D'T'.\frac{x'^2}{a'^4} = 1 - \frac{y'^2}{b'^2}, \text{ or } DT.D'T' = a'^2 = CP^2.$$

61. The greatest ellipse that can be inscribed in a quadrilateral that has two sides h, k, parallel to one another, will

touch those parallels in their middle points; and its area will $= \frac{1}{4}\pi l\sqrt{hk}$, l being the perpendicular distance of the parallels from one another.

Join the two points of contact of one of the inscribed ellipses with the parallel sides by the line DD', (fig. 100) bisect it in C, and through C draw PP' parallel to DT; then C is the centre, and PP', DD', are conjugate diameters of one of the ellipses inscribed in the trapezium; and if $CP = a$, $CD = b$, $DT = c$, $D'T' = d$, $a^2 = cd = (h-c) \times (k-d)$ (Prob. 60); $\therefore kc = h(k-d)$. Now $(\text{area})^2$ of ellipse $\propto a^2 \propto (k-d)\,d \propto \frac{1}{4}k^2 - (\frac{1}{2}k - d)^2$; therefore, for maximum area, $d = \frac{1}{2}k$, and $c = \frac{1}{2}h$; and maximum area $= \pi a \cdot \frac{1}{2}l = \frac{1}{4}\pi l\sqrt{hk}$.

If $h = k$, the trapezium becomes a parallelogram, and the greatest ellipse $= \frac{1}{4}\pi \times$ area of parallelogram.

62. In a given parallelogram to inscribe an ellipse of given eccentricity.

Every ellipse must have its centre in the intersection of the diagonals; and as in the preceding Problem, if Q be the middle point of the side RT, and $CQ = l$, $QT = k$, $QD = x$, $PCQ = \theta$, $PCD = \omega$, $CP = a$, $CD = b$, then

$$a^2 = DT \cdot D'T' = k^2 - x^2,$$

$$b^2 = l^2 + 2lx\cos\theta + x^2, \quad b\sin\omega = l\sin\theta.$$

If therefore α and β be the semi-axes of the ellipse

$$\alpha^2 + \beta^2 = k^2 + l^2 + 2lx\cos\theta,$$

$$\alpha\beta = l\sin\theta\sqrt{k^2 - x^2};$$

$$\therefore l\sin\theta\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) = \frac{k^2 + l^2}{\sqrt{k^2 - x^2}} + 2l\cos\theta \cdot \frac{x}{\sqrt{k^2 - x^2}},$$

the equation for determining x, since $\dfrac{\beta}{\alpha} = \sqrt{1 - e^2}$ is given.

Since the ratio of β to α is zero when the ellipse coincides with either of the diagonals, it must admit of a maximum; and the corresponding value of x may be obtained from the equation

$$(k^2 + l^2)\, x + 2lk^2 \cos\theta = 0, \quad (1)$$

which determines the ellipse that approaches nearest to a circle of all those that can be inscribed in a given parallelogram.

If the parallelogram be equilateral, and if PD, CT, be joined, then CT bisects both the chord PD, and the angle T, and therefore bisects the chord perpendicularly; therefore CT is an Axis of the curve; and if CY be a perpendicular from C on RT, since $YCT = \frac{1}{2}(\pi - \theta)$,

$$CY = k \sin\theta = a\sqrt{1 - e^2 \cos^2 \tfrac{1}{2}\theta},$$

which gives a. If $e = 0$, QD evidently equals $k\cos\theta$ which agrees with (1) when $l = k$.

63. To inscribe an ellipse in a semi-circle, which shall have a given major axis parallel to the diameter of the semicircle.

$CN = x$, $NP = y$, the co-ordinates of P (fig. 107); then because the normal at P passes through the extremity of the minor axis,

$$y = \frac{b^3}{a^2 - b^2}; \text{ but } (y + b)^2 + x^2 = BP^2 = r^2,$$

$$\text{or } (y + b)^2 + \frac{a^2}{b^2}(b^2 - y^2) = r^2, \; \therefore a^4 = r^2(a^2 - b^2),$$

$$\text{or } b = \frac{a}{r}\sqrt{r^2 - a^2}.$$

Hence in order that the area may be the greatest possible, we must have ab a maximum, or $a^2\sqrt{r^2 - a^2}$ a maximum;

$$\therefore a = r\sqrt{\frac{2}{3}}, \text{ and greatest area} = \frac{2\pi r^2}{3\sqrt{3}}.$$

64. **To find the locus of the middle point of a straight line that always has its extremities on the circumferences of two equal circles given in position.**

A, D, (fig. 108), the centres of the circles, O the middle point between them the origin; $ON = x$, $NP = y$ the co-ordinates of P the middle point of BC; x', y'; x'', y'', the co-ordinates of B and C; $BC = 2c$, $OA = OD = b$, $AB = CD = a$; then

$$(1) \quad 2y = y' + y'',$$

$$(2) \quad 2x = x' - x'',$$

$$(3) \quad 4c^2 = (y' - y'')^2 + (x' + x'')^2,$$

$$(4) \quad a^2 = y'^2 + (x' - b)^2,$$

$$(5) \quad a^2 = y''^2 + (x'' - b)^2,$$

between which five equations we have to eliminate x', y', x'', y''; subtracting (4) and (5) and reducing by means of (1) and (2), we get

$$(6) \quad y(y' - y'') + x(x' + x'') = 2bx.$$

Again adding (4) and (5) and doubling

$$2(y'^2 + y''^2) + 2(x'^2 + x''^2) - 4b(x' + x'') = 4(a^2 - b^2),$$

$$\text{but } (y' + y'')^2 + (x' - x'')^2 = 4(x^2 + y^2)$$

from (1) and (2); therefore, subtracting, and reducing by (3),

$$(7) \qquad x' + x'' = \frac{1}{b}(b^2 + c^2 - a^2 + x^2 + y^2),$$

$$\text{also from (6) } y' - y'' = \frac{2bx}{y} - \frac{x}{y}(x' + x'')$$

$$= -\frac{x}{by}(x^2 + y^2 - b^2 + c^2 - a^2);$$

$\therefore$ substituting in (3), and reducing, we get for the equation required

$$x^2(x^2 + y^2 + c^2 - a^2 - b^2)^2 + y^2(x^2 + y^2 + c^2 - a^2 + b^2)^2 = 4b^2c^2y^2.$$

65. To find the locus of the intersection of two normals to an ellipse at right angles to one another.

Let m be the tangent of the angle which a normal to the ellipse makes with the major axis, then its equation is (Art. 125),

$$(y - mx)\sqrt{a^2 + m^2b^2} + (a^2 - b^2)\,m = 0,$$

and changing m into $-\dfrac{1}{m}$, the equation to a normal perpendicular to it, is

$$(my + x)\sqrt{m^2a^2 + b^2} - (a^2 - b^2)\,m = 0,$$

and we have to eliminate m between these equations.

We get by addition,

$$\frac{a^2 + m^2b^2}{m^2a^2 + b^2} = \frac{(my + x)^2}{(y - mx)^2} \quad (1);$$

$$\therefore\ \frac{a^2 + b^2}{x^2 + y^2} = \frac{a^2 + m^2b^2}{(my + x)^2} = \frac{(a^2 + m^2b^2)(m^2a^2 + b^2)}{(a^2 - b^2)^2m^2};$$

$$\therefore\ \frac{a^2 + b^2}{x^2 + y^2}(a^2 - b^2)^2 = (a^2 + b^2)^2 + a^2b^2\left(m - \frac{1}{m}\right)^2 \quad (2).$$

But from (1) we get

$$\frac{a^2 - b^2}{a^2 + b^2} = \frac{x^2 - y^2}{x^2 + y^2} + \frac{4mxy}{(x^2 + y^2)(1 - m^2)};$$

$$\therefore\ \frac{1}{m} - m = \frac{2xy\,(a^2 + b^2)}{a^2y^2 - b^2x^2},$$

which value substituted in (2) after reduction gives the required equation

$$\frac{(a^2 - b^2)^2}{(a^2 + b^2)(x^2 + y^2)} = \left(\frac{a^2y^2 + b^2x^2}{a^2y^2 - b^2x^2}\right)^2.$$

66. To find the locus of a point from which if four normals be drawn to a curve of the second order, the sum of their squares shall be constant.

Let $y^2 + nx^2 = c^2$ (1) be the equation to the curve, and (a, b) the point whose locus is required; the equation to the normal through it is

$$y - b = \frac{y}{nx}(x - a), \therefore y = \frac{nbx}{(n-1)x + a};$$

$$\therefore \left\{\frac{nbx}{(n-1)x + a}\right\}^2 + nx^2 = c^2;$$

$$\text{or } x^4 + \frac{2a}{n-1}x^3 + \frac{n(a^2 + nb^2) - (n-1)^2 c^2}{n(n-1)^2}x^2 + \&c. = 0.$$

Similarly,

$$y^4 - \frac{2nb}{n-1}y^3 + \frac{n(a^2 + nb^2) - (n-1)^2 c^2}{(n-1)^2}y^2 + \&c. = 0.$$

Hence

$$\Sigma(x^2) = \frac{4a^2}{(n-1)^2} - \frac{2\{n(a^2 + nb^2) - (n-1)^2 c^2\}}{n(n-1)^2},$$

$$\Sigma(-2ax) = -2a\left(\frac{-2a}{n-1}\right) = \frac{4a^2}{n-1},$$

$$4a^2 = 4a^2,$$

$$\Sigma(y^2) = \frac{4n^2 b^2}{(n-1)^2} - \frac{2\{n(a^2 + nb^2) - (n-1)^2 c^2\}}{(n-1)^2},$$

$$\Sigma(-2by) = -2b\left(\frac{2nb}{n-1}\right) = \frac{-4nb^2}{n-1},$$

$$4b^2 = 4b^2,$$

$$\therefore \Sigma\{(x-a)^2 + (y-b)^2\}$$

$$= a^2\left\{\frac{4}{(n-1)^2} - \frac{2}{(n-1)^2} + \frac{4}{n-1} + 4 - \frac{2n}{(n-1)^2}\right\}$$

$$+ b^2\left\{\frac{4n^2}{(n-1)^2} - \frac{2n^2}{(n-1)^2} - \frac{4n}{n-1} + 4 - \frac{2n}{(n-1)^2}\right\}$$

$$+ c^2\left(\frac{2}{n} + 2\right) = 4R^2 \text{ a constant};$$

$$\text{or } a^2\left(\frac{4n-2}{n-1}\right) + b^2 \cdot \left(\frac{2n-4}{n-1}\right) = 4R^2 - 2c^2\left(1 + \frac{1}{n}\right);$$

$$\text{or } a^2 \cdot (2n-1) + b^2 \cdot (n-2) = (n-1)\left\{2R^2 - c^2\left(1 + \frac{1}{n}\right)\right\}.$$

67. If an ellipse be inscribed in a quadrilateral, the lines joining the extremities of either diagonal with the points of contact will intersect in the other diagonal.

In fig. 109, because the sides of the triangle KAC are cut by a straight line in the points N, I, M,

$$\frac{KN}{KM} = \frac{NC}{CI} \cdot \frac{AI}{AM};$$

$$\text{but} \quad \frac{KM}{KN} = \frac{CR}{DR} \cdot \frac{MD}{CN}$$

because the ellipse is inscribed in the triangle KCD,

$$\therefore\ CR \,.\, MD \,.\, AI = DR \,.\, AM \,.\, IC,$$

which proves that if CM and AR be joined they will intersect in DI. Also, because the quadrilateral is circumscribed about the ellipse,

$$\frac{AI}{IC} = \frac{DR \,.\, AM}{CR \,.\, DM} = \frac{AL \,.\, BN}{BL \,.\, NC},$$

$$\text{or} \quad BL \,.\, NC \,.\, AI = AL \,.\, BN \,.\, IC,$$

which proves that AN, CL, BD, intersect in a point.

68. If an ellipse be inscribed in a quadrilateral, the line joining its points of contact with two opposite sides passes through the intersection of the diagonals.

Let K the point in which the opposite sides BC, AD intersect, be taken for the origin (fig. 109). $KA = a$, $KB = b$, $KC = c$, $KD = d$; and $KN = k$, $KM = h$, M and N being the points of contact in AD, BC; then the equations to BD, AC, and NM are, respectively,

$$\frac{x}{d} + \frac{y}{b} = 1, \quad \frac{x}{a} + \frac{y}{c} = 1, \quad \frac{x}{h} + \frac{y}{k} = 1;$$

and therefore the condition for their passing through a point is

$$\frac{1}{ak} + \frac{1}{bh} - \frac{1}{ab} = \frac{1}{dk} + \frac{1}{ch} - \frac{1}{cd} \quad (1).$$

Now the equation to the ellipse is (Art. 228),

$$\left(\frac{x}{h}-1\right)^2+\left(\frac{y}{k}-1\right)^2+mxy=1;$$

and for its intersection with the line CD whose equation is $\frac{x}{d}+\frac{y}{c}=1$, we have

$$\left(\frac{x}{h}-1\right)^2+\frac{c^2}{k^2}\left(\frac{x}{d}+\frac{k}{c}-1\right)^2+mc\left(x-\frac{x^2}{d}\right)=1;$$

which must be a perfect square since CD is a tangent,

$$\therefore\quad \frac{m}{2}+\frac{1}{hk}=\frac{2}{cd}\left(\frac{c}{k}+\frac{d}{h}-1\right)=\frac{2}{ab}\left(\frac{b}{k}+\frac{a}{h}-1\right),$$

(since $\frac{x}{a}+\frac{y}{b}=1$ is also a tangent), which is the condition (1).

69. To find the area of an ellipse inscribed in a given quadrilateral, and touching one of the sides in a given point.

As in the preceding Problem, taking two of the sides of the quadrilateral for the co-ordinate axes, the equation to the ellipse will be

$$\left(\frac{x}{h}-1\right)^2+\left(\frac{y}{k}-1\right)^2+2lxy=1,\quad (lhk)^2<1;$$

$$\text{where } l=\frac{2}{ab}\left(\frac{a}{h}+\frac{b}{k}-1\right)-\frac{1}{hk},$$

$$\text{or}\quad hkl=1-2\left(\frac{h}{a}-1\right)\left(\frac{k}{b}-1\right);$$

$$\therefore\quad y=k(1-klx)\pm k\left\{2-x\left(lk+\frac{1}{h}\right)\right\}^{\frac{1}{2}}.\left\{\left(\frac{1}{h}-lk\right)x\right\}^{\frac{1}{2}}.$$

Hence when the ordinate becomes a tangent at N', we have

$$x=KT=\frac{2h}{hkl+1}.$$

Therefore when $x = \frac{1}{2} KT$, the radical in the value of y assumes its greatest value $= k \left(\frac{1 - hkl}{1 + hkl}\right)^{\frac{1}{2}} = OQ$ the semi-diameter conjugate to NN';

$$\therefore \quad \text{area of ellipse} = \pi hk \sin\omega \frac{(1 - hkl)^{\frac{1}{2}}}{(1 + hkl)^{\frac{3}{2}}}$$

$$= \frac{1}{2}\pi ab \sin\omega \frac{\left(1 - \frac{a}{h}\right)^{\frac{1}{2}} \left(1 - \frac{b}{k}\right)^{\frac{1}{2}}}{\left(\frac{a}{h} + \frac{b}{k} - 1\right)^{\frac{3}{2}}}.$$

Now let $\frac{AI}{IC} = n$, $\frac{BI}{ID} = m$, $\frac{BL}{AL} = x$, L being the given point where the ellipse is to touch AB;

then, since $\quad BL \, . \, MA \, . \, DI = AL \, . \, MD \, . \, BI$,

$$\frac{AM}{MD} = \frac{m}{x}, \quad \text{and} \quad \frac{AM}{AD} = \frac{m}{m + x}.$$

$$\text{But} \quad \frac{KA}{KD} = \frac{\Delta BAC}{\Delta BDC} = \frac{BI}{BD} \cdot \frac{AC}{IC} = \frac{m(n + 1)}{m + 1};$$

$$\therefore \quad \frac{KA}{AD} = \frac{m(n + 1)}{1 - mn}; \quad \text{hence} \quad \frac{AM}{KA} = \frac{1 - mn}{(n + 1)(m + x)},$$

$$\text{and} \quad \frac{AM}{KM} \quad \text{or} \quad 1 - \frac{a}{h} = \frac{1 - mn}{1 + m + x(1 + n)};$$

$$\text{consequently} \quad 1 - \frac{b}{k} = \frac{(1 - mn)x}{1 + m + x(1 + n)};$$

$$\therefore \quad \text{area} = \frac{1}{2}\pi ab \sin\omega\, (1 - mn)^{\frac{1}{2}} \frac{\{(1 + n)x^2 + (1 + m)x\}^{\frac{1}{2}}}{\{m(n + 1) + (m + 1)nx\}^{\frac{3}{2}}}.$$

70. In a given quadrilateral to inscribe an ellipse whose area shall be the greatest possible.

The expression for the area in the preceding Problem vanishes when $x = 0$, and when $x = \infty$; and also when

$(1+n)z+1+m=0$, which gives $AM=-AK$; corresponding to which values of z the ellipse becomes coincident with the diagonals BD, AC; and with the line joining K, and the point in which AB, CD intersect. The area of the ellipse will consequently be a maximum for some value of z between zero and infinity given by the equation

$$z^2+2z\left(\frac{m+1}{n+1}-\frac{m}{n}\cdot\frac{n+1}{m+1}\right)-\frac{m}{n}=0.$$

The ratio v of CR to RD must be the same function of $\frac{1}{n}$ and $\frac{1}{m}$ that z is of m and n, and is therefore given by the equation $v^2+2v\left(\frac{m}{n}\cdot\frac{n+1}{m+1}-\frac{m+1}{n+1}\right)-\frac{m}{n}=0$; which shews that the negative value of z taken positively is the value of the ratio of CR to RD.

71. In a given quadrilateral to inscribe an ellipse whose axes shall have a given ratio.

By Prob. 69, since the equation to the diameter NN' is $k-y=k^2lx$, we have

$$ON^2=\frac{h^2}{(1+hkl)^2}(1-2k^2l\cos\omega+k^4l^2),$$

$$\text{and}\quad OQ^2=k^2\cdot\frac{1-hkl}{1+hkl};$$

$$\therefore\quad \alpha^2+\beta^2=\frac{h^2+k^2-2h^2k^2l\cos\omega}{(1+hkl)^2},$$

$$\text{and}\quad \alpha\beta=hk\sin\omega\,\frac{(1-hkl)^{\frac{1}{2}}}{(1+hkl)^{\frac{3}{2}}},$$

α and β denoting the semi-axes of the ellipse; also let v denote their ratio,

$$\therefore\quad \sin\omega\left(v+\frac{1}{v}\right)hk\left\{1-(hkl)^2\right\}^{\frac{1}{2}}=h^2+k^2-2h^2k^2l\cos\omega;$$

$$\text{but}\quad hkl=1-\frac{2(1-mn)^2z}{(m+1)(n+1)(m+z)(1+nz)}$$

since $\frac{h}{a} - 1 = \frac{1 - mn}{(n+1)(m+z)}$ and $\frac{k}{b} - 1 = \frac{(1-mn)z}{(m+1)(1+nz)}$;

also $\frac{h}{k} = \frac{a}{b} \cdot \frac{m+1}{n+1} \cdot \frac{1+nz}{m+z}$;

and upon substituting these values there arises an equation of the fourth degree for determining z.

72. To find the locus of the middle points of a system of parallel straight lines, each of which joins two points in two given curves.

Let $f(x, y) = 0$, $\phi(x, y) = 0$, $y = mx + c$, be the equations to the two curves, and to one of the chords; transfer the origin to (x', y') the middle point of the chord; then the equations to the curves become

$$f(x' + x,\ y' + y) = 0, \quad \phi(x' + x,\ y' + y) = 0,$$

and the equation to the chord $y = mx$; and if in the former we substitute mx for y, the resulting equations will give values for x, being the abscissæ of the points of intersection of the chord and the curves; and if $+x$ satisfy one of the equations, the other must be satisfied by $-x$; therefore the equation to the locus of the middle points of the chords will result from eliminating x between

$$f(x' + x,\ y' + mx) = 0 \text{ and } \phi(x' - x,\ y' - mx) = 0.$$

Suppose the curves to be a hyperbola and its conjugate; the result of the elimination will be found to be

$$\tfrac{1}{4}a^4b^4(a^2m^2 - b^2) = (a^2my' - b^2x')^2(b^2x'^2 - a^2y'^2).$$

If the asymptotes be taken for axes, the result will be

$$x'y'(mx' + y')^2 + mc^4 = 0.$$

73. To find the locus of the vertex of a triangle, upon a given base, and having its vertical angle bisected by a line parallel to a given line.

Take the given line for the axis of y, the middle point of the base $(2a)$ for the origin, and let the angle between them $= \alpha$; then $AM = x \operatorname{cosec} \alpha$, $RN = a \cos \alpha$; (fig. 110);

therefore $\dfrac{a + x \operatorname{cosec} \alpha}{a - x \operatorname{cosec} \alpha} = \dfrac{BM}{MC} = \dfrac{BP}{PC} = \dfrac{PR}{PS} = \dfrac{y - a \cos \alpha}{y + a \cos \alpha}$,

$$\therefore \quad xy = \tfrac{1}{2} a^2 . \sin 2\alpha.$$

74. **To find the locus of the intersection of the tangent to a given curve, and the perpendicular let fall upon it from a given point.**

Let $y - y' = \tan \alpha \, (x - x')$ be the equation to the tangent to a curve at a point (x', y'); then $\tan \alpha = f(x', y')$ is known from the nature of the curve; and the equation to a perpendicular to the tangent from a point (h, k) is $(y - k) \tan \alpha + x - h = 0$; between which three equations, and the equation to the curve if x', y', and $\tan \alpha$ be eliminated, we shall obtain the required relation between x and y.

Thus the equations to the tangent to a parabola and to a line perpendicular to it from (h, k) being

$$y = mx + \frac{a}{m}, \quad y - k = -\frac{1}{m}(x - h);$$

the equation to the locus of their intersection is

$$a(y - k)^2 + x(x - h)^2 + y(y - k)(x - h) = 0;$$

which, if $h = k = 0$, becomes $y^2 (a + x) + x^3 = 0$ the equation to the Cissoid of Diocles; and if $k = 0$, $h = a$, it becomes

$$x \{y^2 + (x - a)^2\} = 0, \text{ or } x = 0,$$

the equation to the tangent at the vertex.

Similarly, the equation to the tangent to the ellipse being $y - mx = \sqrt{b^2 + m^2 a^2}$, the locus of its intersection with a perpendicular let fall from a point (h, k) has for equation

$$y(y - k) + x(x - h) = \{a^2 (x - h)^2 + b^2 (y - k)^2\}^{\frac{1}{2}};$$

which, if the perpendicular be dropped from the centre, becomes

$$(x^2 + y^2)^2 = a^2 x^2 + b^2 y^2,$$

which agrees with the polar equation already found (Art. 135)

$$r^2 = a^2 (1 - e^2 \sin^2 \theta).$$

In the case of the hyperbola, changing b^2 into $-b^2$, the equation is

$$(x^2+y^2)^2 = a^2x^2 - b^2y^2,$$

which if $b = a$ becomes

$$(x^2+y^2)^2 = a^2(x^2-y^2),$$

representing a curve called the Lemniscata of Bernouilli, and whose polar equation is $r^2 = a^2 \cos 2\theta$. If the perpendicular be let fall from the vertex of a rectangular hyperbola, the equation is $x^2 + y^2 = a(x - \sqrt{x^2 - y^2})$.

75. If a curve roll upon an equal one, similar points being always in contact, to find the locus of any given point in the rolling curve.

Let AP be the fixed, and $A'P$ the rolling curve (fig. 71), S' the describing point, and S a point similarly situated in the fixed curve. Join SS' meeting the common tangent at P in Y; also join SP, $S'P$. Then because the points in contact are similar, SP, $S'P$ are equal and equally inclined to the common tangent PY; therefore PY bisects SS' at right angles, and therefore the locus of S' is similar to that of Y, the foot of the perpendicular from S upon the tangent to the fixed curve, and $S'P$ is always a normal to the locus of S'.

If therefore $y = f(x)$ be the equation to the locus of Y, $\frac{y}{2} = f\left(\frac{x}{2}\right)$ is the equation to the locus of S'. Hence if the curves are equal parabolas, and S the focus, its locus will be a straight line; if S be the vertex, its locus will be the Cissoid of Diocles; if the curves are ellipses, and S the focus, its locus will be a circle; if S be the centre, the equation to its locus will be

$$x^2 + y^2 = 2\sqrt{a^2x^2 + b^2y^2}.$$

76. To trace the curve $y^2 = \dfrac{ax(x-3a)}{x-4a}$.

When $x = 0$, $y = 0$, and limit of $\frac{y}{x} = \infty$; therefore at the origin the curve cuts the axis of x at right angles. When

$x = 3a$, $y = 0$, and limit of $\dfrac{y}{x - 3a}$, when $x = 3a$, is infinity, therefore the curve again cuts the axis at right angles at a distance $3a$ from the origin. For values of x between $3a$ and $4a$, y is impossible, and there is no curve; when $x = 4a$, y is infinite; and when x is very large, the relation between x and y becomes

$$y^2 = ax\left(1 - \frac{3a}{x}\right)\left(1 - \frac{4a}{x}\right)^{-1} = ax\left(1 + \frac{a}{x} + \frac{4a^2}{x^2}\right),$$

$\therefore$ $y^2 = a(x + a)$ is the equation to the parabolic asymptote, above which the curve lies. There is a maximum ordinate when $x = 2a$, and a minimum ordinate when $x = 6a$; and there must evidently be a point of contrary flexure at G. By taking the limiting value of $\dfrac{y^2}{x}$, it will be found that the diameter of curvature at $A = \dfrac{3a}{4}$, and similarly at C it will be found to equal $3a$. There is no part of the curve corresponding to negative values of x; and the axis of x is an Axis of the curve. Hence the curve is such as is represented in fig. 111, the dotted line being intended for the parabolic asymptote.

77. To trace the curve

$$(y^2 - x^2)(x - 1)(x - \tfrac{3}{2}) = 2\{y^2 + x(x - 2)\}^2.$$

Solving the equation relative to y^2, we find

$$2y^2 = -\tfrac{3}{2}(x^2 - \tfrac{11}{6}x - \tfrac{1}{2}) \pm \frac{\sqrt{6}}{4}(x - 1)\sqrt{(-x + \tfrac{3}{2})(10x + 1)}.$$

Hence x must lie between $-\frac{1}{10}$ and $\frac{3}{2}$; and as the rational part of $2y^2 = \frac{3}{2}(x + \frac{1}{4})(2\frac{1}{12} - x)$ nearly, when $x = -\frac{1}{10}$ the ordinate is real, and is a tangent to the curve; when $x = 0$, $2y^2 = \frac{3}{4}$ or $= 0$; when $x = 1$, $y^2 = 1$; and when $x = \frac{3}{2}$, we get a real value for the last ordinate, touching the curve; which consequently is that represented in fig. 112, having three true double points, and four double tangents, i. e. straight lines touching it in two points.

78. To trace the curve $y^2 = \dfrac{x^3 - a^3}{x - 2a}$.

$$\text{When} \quad x = 0, \; y = \pm \frac{a}{\sqrt{2}};$$

when $x = a$, the curve cuts the axis of x at right angles; from $x = a$ to $x = 2a$, y remains impossible; when $x = 2a$, y is infinite; and when x and y become very great, the relation between them is

$$\pm \frac{y}{x} = \left(1 - \frac{a^3}{x^3}\right)^{\frac{1}{2}} \left(1 - \frac{2a}{x}\right)^{-\frac{1}{2}}$$

$$= \left(1 - \frac{a^3}{2x^3}\right)\left(1 + \frac{a}{x} + \frac{3a^2}{2x^2}\right) = 1 + \frac{a}{x} + \frac{3a^2}{2x^2};$$

$\therefore \pm y = x + a$ is the equation to the asymptotes above which the curve lies; also when x is negative, y perpetually increases, and there are two infinite branches; therefore the curve is such as is represented in (fig. 113), the co-ordinates of the points P, P' where it cuts the asymptotes being

$$x = -\frac{a}{3}, \; y = \pm \frac{2a}{3}.$$

79. The corner of a page is turned down so that the triangle is of a constant area, a^2; the locus of the angular point is a lemniscata whose equation is $r^2 = a^2 \sin 2\theta$.

80. If two circles be inscribed in another circle touching one another, then the area of the circle whose diameter is their common tangent, will equal the area between the greater semicircle and the two smaller ones.

81. The equation $y^4 + 2a^2xy - x^4 = 0$ expressed by Polar co-ordinates is $r^2 = a^2 \tan 2\theta$.

82. Of the three squares that can be inscribed in an acute-angled triangle, the greatest is that which has two angles in the least side.

83. A parabola is bounded by an ordinate perpendicular to its axis, whose length is b, that of the portion of the

axis cut off being a; D, d are the diameters of the circumscribed and inscribed circles, then $D + d = a + b$.

84. In PG the normal to an ellipse, a point Q is taken such that $PQ = CD$, shew that Q traces out a circle.

85. Two conjugate diameters are produced to intersect the same directrix of an ellipse, and from the point of intersection of each, a perpendicular is drawn to the other; these perpendiculars will intersect in the nearer focus.

86. If a pair of conjugate diameters of an ellipse when produced, be asymptotes to a hyperbola, the point of the hyperbola at which the tangent will also touch the ellipse, lies in an ellipse similar to the original one.

87. If two given circles touch one another internally, and a series of circles whose radii are r_1, r_2, &c. be described between them touching one another; and if P_1, P_2, &c. be the perpendiculars dropped from their centres upon the common diameter of the given circles, then $\dfrac{P_n}{r_n} - \dfrac{P_1}{r_1} = 2(n-1)$.

88. If a, b, r, be respectively the radii of the given circles, and of the first circle in the series, prove that the radius of the $(n+1)^{\text{th}}$ circle will be

$$= \frac{ab(a-b)r}{abr + \{n(a-b)\sqrt{r} \pm \sqrt{ab(a-b-r)}\}^2}.$$

89. If two circles touch one another, the radius of any circle touching them both bears an invariable ratio to the perpendicular from its centre upon their common tangent.

90. If the length of the axis of an oblique cone be equal to the radius of its base, every section perpendicular to the axis will be a circle.

91. If an ellipse be moved between two straight lines at right angles to one another, to shew that the centre will describe a circle, and to find the locus of any given point in the axis.

92. To find the equation to the conic section described with focus (h, k) and directrix $y = mx + c$.

93. If SY, HZ be perpendiculars from the foci upon the tangent at any point P of an ellipse; then SZ and HY will intersect in the middle point of the normal at P, and the locus of their intersection will be an ellipse with $a(1 + e^2)$ and $a\sqrt{1 - e^2}$ for axes.

94. If a parabola be moved between two straight lines at right angles to one another, the equation to the locus of its vertex will be $x^{\frac{4}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}}x^{\frac{2}{3}} = a^2$.

95. The area between two normals to a parabola at the extremities of a focal chord, and the curve, $= \dfrac{20a^2}{3\sin^3 2\theta}$, θ being the inclination of one of the normals to the axis.

96. The sum of the squares of the normals to an ellipse drawn at the extremities of conjugate semi-diameters

$$= a^2(e^2 - 1)(e^2 - 2).$$

97. Find the locus of the vertex of a triangle whose base is constant, and likewise the product of the perpendiculars dropped from the extremities of the base upon the line bisecting the vertical angle.

98. If P be a point in a hyperbola, whose ordinate $= BC^{\frac{3}{2}} \div \sqrt{SC}$, and CY be a perpendicular from the centre upon the tangent at P, then $PY = SC$.

99. If the opposite sides of a hexagon inscribed in a conic section be produced to meet, the three points of intersection will lie in a straight line.

In fig. 114, draw any diagonal MM', and let the pairs of opposite sides which pass through its extremities meet in C, B; and taking the line CB for the axis of x, let the equation to the conic section be

$$ay^2 + bxy + cx^2 + dy + ex + f = 0 \qquad (1),$$

and let the equations to the sides $M'M$, MN, NN', $N'M'$ of one of the quadrilaterals into which MM' divides the hexagon, be respectively

$$px + p'y = 1, \qquad qx + q'y = 1,$$

$$rx + r'y = 1, \qquad sx + s'y = 1.$$

Now the equation to the conic section is satisfied by all such values of x and y as jointly satisfy the equations to any two adjacent sides of the quadrilateral; and therefore its equation, since it is of the second degree, must be of the form

$$m(px+p'y-1)(rx+r'y-1)+n(qx+q'y-1)(sx+s'y-1)=0,$$

which compared with (1) gives

$$mpr + nqs = c, \quad m + n = f,$$

$$m(p+r) + n(q+s) + e = 0.$$

Now if we suppose A, B, C to be given points, and therefore p, q, s to be given quantities, these three equations determine m, n and r; therefore D is a fixed point; which shews that if three sides of a quadrilateral inscribed in a conic section pass through three fixed points in a given straight line, the remaining side also will pass through a fixed point in that line. Consequently, since three sides of the quadrilateral $MOO'M'$ pass through the points A, B, C, the remaining side OO' must also pass through D; therefore the three intersections of the opposite sides of the hexagon lie in a straight line. If the hexagon be changed into a triangle, by supposing every other side to become evanescent, and therefore to assume the direction of a tangent to the conic section at one of the angular points of the triangle, we fall upon Prob. 50.

100. If two pairs of opposite sides of a hexagon inscribed in a conic section be parallel to one another, the two remaining sides shall also be parallel to one another.

Let MM' be any diagonal of any hexagon inscribed in a conic section having two pairs of opposite sides parallel to one

another, and as in the preceding Problem, let the equations to the four sides of the quadrilateral $M'MNN'$ be

$$y + px + p' = 0, \qquad y + qx + q' = 0,$$

$$y + rx + r' = 0, \qquad y + sx + s' = 0;$$

then the equation to the conic section will be

$$m(y+px+p')(y+rx+r') + n(y+qx+q')(y+sx+s') = 0,$$

which compared with (1) in the preceding Problem, gives

$$m + n = a, \qquad mpr + nqs = c,$$

$$m(r+p) + n(s+q) = b.$$

These equations shew that if three of the quantities p, q, r, s be given, the fourth is also constant; i.e. if three of the sides of any quadrilateral inscribed in a conic section be parallel to three given straight lines, the remaining side is also parallel to a fixed line. But if $O'M'$, OM be respectively parallel to the lines to which MN, $M'N'$ are parallel, then the position of OO' is determined from the above equations by interchanging q and s which does not alter them; therefore OO' is parallel to the same line to which NN' is parallel; or the two remaining sides of the hexagon are parallel to one another.

101. The three diagonals of a hexagon circumscribed about a conic section intersect in a point.

Let $\alpha\beta\gamma\delta\epsilon\kappa$ (fig. 115) be the angular points of a hexagon circumscribed about a conic section; join the points of contact by straight lines, so forming the inscribed hexagon $ABCDEK$; and produce its opposite sides to meet in P, Q, R. Then if two tangents were applied at the points α', δ', in which the diagonal $\alpha\delta$ meets the curve, they would intersect in P; similarly the pairs of tangents applied at the points where $\gamma\kappa$, $\beta\epsilon$, meet the curve, would respectively intersect in Q and R; but P, Q, R, lie in a straight line; therefore the three diagonals (since they are in the directions of chords joining the points of contact of pairs of tangents drawn from points in a straight line) must (Art. 50) pass through the same point.

SECTION XI.

ON CURVES OF THE THIRD AND FOURTH AND HIGHER ORDERS; AND ON THE SINGULAR POINTS OF CURVE LINES.

250. IN this section we shall give some of the principal results that have been obtained relative to the properties of curves of the 3rd and 4th orders; and as Plücker, to whom the following investigations are chiefly due, has applied his method to curves of the second order, as well as of higher orders, we shall commence with that application; both for the sake of some new results to which it leads, and for the purpose of making Plücker's general method more readily understood.

251. The general equation of the second degree

$$y^2 + 2Axy + Bx^2 + 2Cy + 2Dx + E = 0, \qquad (1),$$

provided $x^2 + 2Ax + B = 0$ has not equal roots, can always be transformed into

$$(y + ax + b)(y + a'x + b') + m = 0, \qquad (2);$$

only when the auxiliary equation has imaginary roots, the factors of the transformed equation are likewise imaginary, but their product real.

As equation (2) is of the 2nd degree, and contains the requisite number of independent constants, we may evidently assume it to be identical with (1); and upon expanding and equating coefficients, we find

$$a + a' = 2A, \quad aa' = B;$$

so that $-a$, $-a'$, are the roots of $x^2 + 2Ax + B = 0$, and will be real provided $A^2 - B > 0$. To determine b, b', and m, we have

$$b + b' = 2C, \ ab' + a'b = 2D, \ bb' + m = E, \qquad (3);$$

therefore if a, a' are real, these equations will evidently furnish a single system of real values of b, b', and m. But if a, a', are imaginary, i.e. if $A^2 - B < 0$, the equation

$$ab' + a'b = 2D$$

shews that b, b' must also be conjugate imaginary roots of a quadratic equation, and their product consequently real, and therefore m real; and in this case the two factors of the transformed equation are imaginary, but their product (which equals $-m$) is real. Hence the proposed transformation can be effected, and only in one way; none of the coefficients being indeterminate, nor having more than one value.

252. If a and a' are equal, then $A^2 - B = 0$; and equations (3) become inconsistent with one another, unless

$$D = aC = AC,$$

when the two former of them become identical. Hence when

$$A^2 - B = 0,$$

the transformation (2) is impossible, and it may be replaced by

$$(y + ax + b)^2 + m(y + cx + d) = 0,$$

in which one constant may be assumed at pleasure (since the general equation, with the condition $A^2 - B = 0$, contains but four independent constants), and then all the others can be determined from linear equations.

If from the two former of equations (3) we determine b, b', and substitute them in $bb' + m = E$, we get

$$m(A^2 - B) = D^2 - 2ADC + BC^2 + E(A^2 - B);$$

and if the second member of this equation vanish, the proposed equation resolves itself into two factors of the first degree, and represents two straight lines; if the second member be negative at the same time that $A^2 - B$ is negative, the proposed equation cannot be satisfied by any real values of x and y.

253. When the general equation to a curve of the second order is put under the form

$$(y + ax + b)\,(y + a'x + b') + m = 0,$$

its two real or imaginary asymptotes have for equations

$$y + ax + b = 0, \quad y + a'x + b' = 0.$$

If with the equation of the second degree, which as we have just shewn may be written $pq + m = 0$, where p and q denote linear functions of x and y each containing two constants, we combine the equation to any straight line, we shall usually determine two points of intersection; but if we take the equation $p = 0$, we get $\frac{m}{q} = 0$, which can only be satisfied by supposing x and y to be infinitely great; so that $p = 0$ represents a straight line whose two intersections with the curve are at an infinite distance, or it is the equation to one of the asymptotes of the curve; and in the same way it appears that $q = 0$ is the equation to the other asymptote. The curve is a hyperbola when the asymptotes are real, and an ellipse in the contrary case; and instead of being determined by five constants as in the case when it is referred to co-ordinate axes whose relation to it is arbitrary, it is, when represented by the equation $pq + m = 0$, determined by one constant, and two straight lines that bear a fixed relation to it.

254. In the second form to which we have reduced the general equation of the second order, viz. $p^2 + mq = 0$, the curve represented is a parabola, and is determined by one constant, and two straight lines, one of which is arbitrary since one constant more than necessary enters into the equation. It is evident that $p = 0$ is the equation to a diameter intersecting $q = 0$ at a point in the curve; and that $q = 0$ is the equation to the tangent at that point, as it leads to $p^2 = 0$, shewing that its two points of intersection with the curve coincide with one another.

255. When we take for the general equation of the second degree the form

$$pq + mr^2 = 0,$$

which contains seven constants, and combine it with either of

the equations $p = 0$ or $q = 0$, we get $r^2 = 0$; so that the two straight lines represented by $p = 0$, $q = 0$, are tangents, and the two points of contact lie in the straight line $r = 0$; and the curve is in this case determined by any two tangents and the chord joining the points of contact.

Another form under which the general equation of the second degree may be written, is

$$u^2 + \lambda v^2 = \mu r^2,$$

u, v, and r being, as before, linear functions of x and y, each involving two constants; and in this case each of the lines $u = 0$, $v = 0$, $r = 0$, represents the chord joining the points of contact of the pair of real or imaginary tangents passing through the intersection of the other two lines.

256. The general equation of the third degree

$$y^3+Ay^2x+Byx^2+Cx^3+Dy^2+Eyx+Fx^2+Gy+Hx+I=0 \quad (1),$$

provided $z^3 + Az^2 + Bz + C = 0$ has no equal roots, can always be transformed into

$$(y + ax + b)(y + a'x + b')(y + a''x + b'') + m(y + cx + d) = 0 \quad (2);$$

only when the auxiliary cubic has imaginary roots, two factors of the transformed equation are likewise imaginary, but their product real.

As equation (2) is of the third degree, and contains the requisite number of independent constants, we may evidently assume it to be identical with (1); and upon expanding and equating coefficients we find

$$a + a' + a'' = A, \quad aa' + aa'' + a'a'' = B, \quad aa'a'' = C,$$

so that $-a$, $-a'$, $-a''$, are the roots of the cubic equation

$$z^3 + Az^2 + Bz + C = 0 \quad (3),$$

which we will suppose to be unequal.

For determining b, b', b'', we get three equations, which may be written

$$b + b' + b'' = D, \quad a'b'' + a''b' + a(b' + b'') + (a' + a'')b = E,$$
$$a(a'b'' + a''b') + a'a''b = F \quad (4).$$

Now if a, a', a'', be real and unequal, from these equations since they are linear, a single system of determined values of b, b', b'', can be at once obtained. But if two roots of (3) a', a'', be imaginary, then $a' + a''$, and $a'a''$, are real; and therefore from the same equations we can get one real system of determined values of

$$a'b'' + a''b', \; b' + b'', \text{ and } b;$$

consequently b' and b'' must be conjugate imaginary roots of a quadratic equation since a', a'', are so; therefore $b'b''$ is real; as is also the product

$$(y + a'x + b')(y + a''x + b'').$$

The three remaining constants m, c, and d, are given by the equations

$$b(b' + b'') + b'b'' + m = G, \quad ab'b'' + b(a'b'' + a''b') + mc = H,$$

$$bb'b'' + md = I, \quad (5),$$

which, subject only to the condition of a, a', a'', being unequal, give one real system of determinate values of m, c, d. Hence it is proved that the proposed transformation can be effected, and only in one way; none of the constants of the transformed equation being indeterminate, and none of them having more than one value.

257. If $a' = a''$, equations (4) become

$$b + (b' + b'') = D, \quad (a + a')(b' + b'') + 2a'b = E, \quad aa'(b' + b'') + a'^2 b = F,$$

and cannot coexist unless the equation

$$Da'^2 - Ea' + F = 0$$

is satisfied, in which case they will determine only b, and $b' + b''$; so that the transformation into (2) may be effected in an infinite number of ways, as one of the quantities b', b'', may be assumed at pleasure; and the remaining constants become known from equations (5). When the equation $Da'^2 - Ea' + F = 0$ is not satisfied, it is impossible to put

the proposed equation into the form (2); and in place of it we may choose the form

$$(y+ax+b)\{(y+a'x+b')^2+l(y+ax+b)\}+m(y+cx+d)=0,$$

containing eight constants to which the number of constants in the proposed is reduced on account of the condition $a'=a''$; and by comparing coefficients it will be found that this transformation can be effected in one determined way and no more.

When $a=a'=a''$, equations (4) cannot coexist unless $E=2Da$, $F=Da^2$, in which case they will determine only the value of $b+b'+b''$. Consequently the transformation into (2) may be effected in an infinite number of ways, as two of the quantities b, b', b'' may be assumed at pleasure. Under these conditions, the proposed equation has its independent constants reduced to five; and besides the form (2) may likewise be made to assume a form involving that number of constants, viz.

$$(y+ax+b)^3+m(y+cx+d)=0.$$

When the conditions $E=2Da$, $F=Da^2$, are not satisfied, it is impossible to put the proposed into the form (2); and instead of it we may choose the form

$$(y+ax+b)^3+l(y+gx+h)^2+m(y+cx+d)=0,$$

containing one more constant than necessary, as the number in the original equation is reduced to seven.

If we take the form

$$(y+ax+b)^3+l(y+ax+b)(y+gx+h)+m=0,$$

containing only six constants, there will be an equation of condition which we find to be $Da^2-Ea+F=0$; subject to which the transformation can be effected in one determined way and no more.

258. A curve of the third order has in general three rectilinear asymptotes by which it is cut in three points lying in a straight line.

It has been shewn that the general equation of the third

degree with certain exceptions can be made to assume the form

$$(y + ax + b)(y + a'x + b')(y + a''x + b'') + m(y + cx + d) = 0,$$

which we may write $pqr + ms = 0$, using p, q, &c. to denote linear functions of x and y each involving two constants.

Now if with the equation $pqr + ms = 0$, we combine the equation to the straight line $p = 0$, we get $s = 0$; so that we determine only one point of intersection of the line $p = 0$ with the curve, namely that in which the line $p = 0$ intersects the line $s = 0$; therefore the other two points of intersection which ought to exist must be at an infinite distance; consequently the line $p = 0$ is an asymptote to the curve.

Similarly, the equations $q = 0$ $r = 0$, represent two rectilinear asymptotes, each having, in common with the curve, but one point; and that point situated in the line $s = 0$. Therefore a curve of the third order has in general three rectilinear asymptotes by which it is cut in three points lying in a straight line.

259. To construct any number of points of a curve of the third order, having given its three rectilinear asymptotes P, Q, R fig. (116); its three points of intersection with those asymptotes S, S_1, S_2; and one other point M.

In this case the equation to the curve will be

$$pqr + ms = 0 \qquad (1),$$

all the constants being known; and by introducing an indeterminate constant n, it may be written

$$p(qr + n) + ms - np = 0.$$

Under this form it is satisfied by

$$qr + n = 0 \quad (2), \qquad ms - np = 0 \quad (3);$$

the former representing a hyperbola that has Q, R, for its asymptotes; and the latter a straight line passing through S the intersection of the proposed curve with the remaining

asymptote, and whose two points of intersection with the hyperbola are situated in the curve (1). Now suppose the hyperbola to pass through M, then n will receive a determined value; and the line (3) will pass through M, and the second point N which it has in common with the hyperbola, and therefore with the curve (1), is determined by taking $LN=MK$, since Q and R are asymptotes to the hyperbola. Similarly, by joining M with S_1 and S_2, two other points may be determined; and any one of these new points will enable us to determine three fresh points of the curve, and so on.

260. If a series of curves of the third order have the same three rectilinear asymptotes, and cut them in the same points, the locus of the points of contact of the tangents passing through any one of those points, will be a hyperbola.

In fig. 116, let the point M be such that it bisects LK, then N coincides with it, and SM is a tangent at M, and does not meet the curve except at M and S. Let $p=0$ be the equation to the asymptote passing through S; and let p_0, p', p'' be the values assumed by the given linear function p, when in it the co-ordinates of S, K, L are substituted for x and y, and P its value when the co-ordinates of M are substituted; and similarly for the linear functions q and r, $q=0$ and $r=0$ being the equations to the other two asymptotes. Then because M lies midway between K and L, we must have between the co-ordinates of those points the relations

$$2X = x' + x'', \quad 2Y = y' + y'';$$

and consequently $2Q = q' + q'' = q''$, since $q' = 0$. Now the equation to SL may be written $q - q_0 = k(r - r_0)$, and substituting in it the co-ordinates of L, we get $q'' - q_0 = -kr_0$;

$$\therefore\ 2Q = q'' = q_0 - kr_0;$$

but $Q - q_0 = k(R - r_0)$, since M lies in SL; therefore, eliminating the indeterminate constant k, we have

$$2QR - Qr_0 - Rq_0 = 0$$

for the equation to the locus of M, which represents a hyperbola passing through S, and through the point of intersection

of the asymptotes Q and R, and having its centre midway between those points, and its asymptotes parallel to the said asymptotes.

261. A curve of the third order has in general three centres whose positions depend only on the asymptotes, and the points in which the curve cuts the asymptotes.

Proceeding as in the last article, we should find for the loci of the points of contact of the tangents passing through the points S_1 and S_2, two other hyperbolas having for equations

$$2pr - pr_1 - rp_1 = 0, \quad 2pq - pq_2 - qp_2 = 0.$$

Now any two of these hyperbolas, having asymptotes parallel to the same straight line, can only intersect in three points; and we shall shew that these three points are fixed points, and the same for the three hyperbolas. For the line SS_2, since it passes through the points S, S_2; and through the points S_1, S_2; may evidently be represented by either of the equations

$$\frac{p}{p_2} + \frac{r}{r_0} = 1, \quad \frac{q}{q_2} + \frac{r}{r_1} = 1,$$

which lead to the two conditions

$$(r_0 - r_1)\, p_2 = r_0 p_1 \qquad (r_1 - r_0)\, q_2 = r_1 q_0 \qquad (1).$$

Now if we eliminate r between the equations to the two first hyperbolas, viz.

$$2qr - qr_0 - rq_0 = 0,$$
$$2pr - pr_1 - rp_1 = 0,$$

we get for the curve passing through their points of intersection the equation

$$2pq - \frac{r_1 q_0}{r_1 - r_0} \cdot p - \frac{r_0 p_1}{r_0 - r_1} \cdot q = 0,$$

which by virtue of the equations of condition (1) is identical with the equation to the third hyperbola

$$2pq - pq_2 - qp_2 = 0;$$

therefore the three hyperbolas intersect one another in the same three points.

These points, which are fixed points for all curves of the third order that have the same asymptotes, and cut them in the same three points S, S_1, S_2, are called the centres of the curves, and are such that three chords drawn through each of them to meet the curve again in S, S_1, S_2, mutually bisect one another.

262. If we suppose the point M with which we begun the construction of the curve, to coincide with one of the fixed centres, then as two of the hyperbolas pass through it, two tangents to the curve will likewise pass through it, or it is a double point. When, therefore, in the equation to curves of the third order $pqr + ms = 0$, that have three given asymptotes and cut them in three given points, we determine m (which is the only constant that remains indeterminate) so that the curve passes through one of its centres, that centre becomes a double point of the curve; consequently a double point can only occupy three different positions.

263. When a straight line cutting a curve of the third order in three points moves parallel to itself, the locus of the point D in it whose distance from its extremity always equals one third the sum of the distances of the three points of intersection with the curve from its extremity, will be a straight line.

Let $q = kp + \gamma$ be the equation to the line in which k is known; and let the equation to the curve be

$$(nx + a)(y + b)(c - nx - y) + m(y + gx + h) = 0,$$

$$\text{or } pq(\beta - p - q) + ms = 0;$$

the co-ordinate axes being taken parallel to two asymptotes, and $a + b + c = \beta$. Then eliminating q, the co-ordinates of the points of intersection of the line and curve must satisfy the equation

$$k(k+1)p^3 + (\gamma + 2k\gamma - k\beta)p^2 + \gamma(\gamma - \beta)p - ms = 0.$$

Now let P and Q be the values assumed by the linear functions p and q, when in them the co-ordinates of D are substituted for x and y, then

$$3P = \frac{k\beta - (2k+1)\gamma}{k(k+1)}; \text{ also } \gamma = Q - kP;$$

hence eliminating γ, we get for the locus of D,

$$k(k+2)P + (2k+1)Q - k\beta = 0 \qquad (1),$$

the equation to a straight line, which is called a diameter of the curve of the third order.

When k changes, the direction of this diameter changes; and if two values of k become equal, i. e. if k, besides satisfying (1), also satisfy the equation

$$(k+1)P + Q - \tfrac{1}{2}\beta = 0,$$

and we eliminate k between this and (1), we shall obtain the equation to the curve to which all the diameters are tangents; viz.

$$(2p + 2q - \beta)^2 = 4pq, \text{ or } (p - \tfrac{1}{2}\beta)^2 + (q - \tfrac{1}{2}\beta)^2$$
$$+ (r - \tfrac{1}{2}\beta)^2 = (\tfrac{1}{2}\beta)^2 \qquad (2).$$

This result shews that the diameters of a curve of the third order having three real asymptotes, are all tangents to the ellipse which touches in their middle points the three sides of the triangle formed by the asymptotes.

For equation (2) shews that the line $p = 0$ meets and touches the ellipse in a point for which $2q = 2r = \beta$; now let X and Y be the co-ordinates of the point lying midway between the points of intersection (x', y'), (x'', y'') of $p = 0$ with the other two asymptotes, then (as in Art. 260) $2X = x' + x''$, $2Y = y' + y''$; therefore $2Q = q' + q'' = q'' = \beta$, $2R = r' = \beta$, since β is the common value of the functions p, q, r, when in each the co-ordinates of the point of intersection of the other two are substituted; therefore (X, Y) is the point of contact of $p = 0$ with the ellipse; and similarly for the other asymptotes.

264. The number of the double points including cusps of a curve of the n^{th} order cannot exceed $\frac{1}{2}(n-1)(n-2)$.

Let x denote the number of double points of any curve of the n^{th} order; and through them and through $\frac{1}{2}p(p+3) - x$ additional points of the curve, let a curve of the p^{th} order be drawn. Then because in each double point two points of intersection of the curves become coincident, and the total

number of those points of intersection cannot exceed np, we must have

$$2x + \tfrac{1}{2}p\,(p + 3) - x \text{ not greater than } np,$$

$$\text{and } \therefore\ x \text{ not greater than } \tfrac{1}{2}p\,(2n - 3 - p);$$

but this last expression will be a maximum when $p = 2n - 3 - p$ or $2p = 2n - 3$; and as p must be an integer, $p = n - 1$, or $v = n - 2$; and consequently we cannot have x greater than $\frac{1}{2}(n - 1)(n - 2)$.

265. A curve of the n^{th} order may in general have $n(n - 1)$ tangents drawn to it through a given fixed point, or parallel to a given line.

Let $f(x, y) = 0$ be the equation to the curve, and $y = mx + c$ the equation to a straight line; then if m and c be such that $f(x, mx + c) = 0$ has a real root twice repeated, that root is the abscissa to the point of contact of the line with the curve; and (Theory of Equations, Art. 60) is also a root of $f'(x, mx + c) = 0$; and if between these equations, which are respectively of n and of $n - 1$ dimensions in x and c, we eliminate c, there will result an equation of $n(n - 1)$ dimensions, whose roots are the abscissæ of the points of contact of all tangents that can be applied to the curve parallel to $y = mx$; consequently the number of such tangents will in general be $n(n - 1)$.

But if the tangents are all to pass through a point (h, k), then we must eliminate m between

$$f\{x, [m(x - h) + k]\} = 0, \quad f'\{x, [m(x - h) + k]\} = 0,$$

and the result will have for its roots the abscissæ of the points of contact, in number $n(n - 1)$ as before. When the given point is in the curve, the tangent at that point must evidently be reckoned twice; and if it be in one of the asymptotes at an infinite distance, the number of tangents that can pass through it, or in other words that can be drawn parallel to the asymptote, must be reduced by two units. Hence four tangents can be drawn to a curve of the third order, parallel to one of its asymptotes. If the given direction be parallel to the tangent at a point of inflexion, or the given point be itself a point of

inflexion, the number of tangents that can be drawn, will be respectively diminished by one and two units.

266. When tangents are applied to a curve of the third order respectively parallel to its three asymptotes, the points of contact lie in a straight line.

As we may assume for the general equation to curves of the third order any equation of the third degree in x and y with nine independent constants, we may take for it the form

$$pqr + ms^2 = 0 \qquad (1);$$

but we cannot, as in the fundamental form where only the simple power of s enters, be certain that this transformation can be effected only in one way. The lines expressed by the linear functions p, q, r, s, now bear new relations to the curve; for if with equation (1) we combine the equation $p = 0$, we get $s^2 = 0$; therefore the line $p = 0$ meets the curve in two points only, and those points are coincident; consequently, its other point of intersection is at an infinite distance; so that the line $p = 0$ is parallel to an asymptote, and also touches the curve in the point where it intersects the line $s = 0$. Similarly, $q = 0$ and $r = 0$ represent lines parallel to the other two asymptotes, and touching the curve in the points where they intersect $s = 0$. Now four different tangents may be applied to the curve parallel to any one of its asymptotes, and the four points of contact may be joined by straight lines with the points of contact of four tangents parallel to a second asymptote by sixteen different straight lines, each of which will pass through the point of contact of a tangent parallel to the third asymptote. Since, therefore, there can be sixteen different systems of tangents parallel to the asymptotes with their points of contact in a straight line, the general equation of the 3rd degree must be capable of being put into the form (1) in the same number of ways.

267. A curve of the n^{th} order has in general $3n(n-2)$ points of inflexion.

Let $f(x, y) = 0$ be the equation to the curve, and

$$y = mx + c$$

the equation to a straight line; then if m and c be such that

$f(x, mx + c) = 0$ has a real root thrice repeated, that root is the abscissa to a point of inflexion; and it is (Theory of Equations, Art. 60) also a root of the equations

$$f'(x, mx + c) = 0, \quad f''(x, mx + c) = 0;$$

and if between these three equations m and c be eliminated, there will result an equation whose roots are the abscissæ of points of inflexion.

Now restoring the value of y, the three equations between which the elimination is to be performed may be written

$$u_n = 0,$$
$$u_{n-1} + mv_{n-1} = 0,$$
$$u_{n-2} + 2mw_{n-2} + m^2 v_{n-2} = 0,$$

where u_n, u_{n-1}, &c. denote functions of n, $n - 1$, &c. dimensions in x and y; eliminating m between the two latter, we get an equation of $3n - 4$ dimensions, viz.

$$u_{n-2} v^2_{n-1} - 2w_{n-2} u_{n-1} v_{n-1} + v_{n-2} u^2_{n-1} = 0;$$

and again eliminating y between this and $u_n = 0$, we finally get an equation of $n(3n-4)$ dimensions in x. Now the curve will have n rectilinear asymptotes, each of which will intersect it in two points infinitely distant, which points have the character of points of inflexion in that the radius of curvature at each is infinitely great; therefore $2n$ points are included in the above, which are not proper points of inflexion; and subtracting, we get $3n(n - 2)$ for the number of points of inflexion that a curve of the n^{th} order may in general have.

268. The points of inflexion of a curve of the third order lie three and three in a straight line.

By the foregoing article a curve of the third order has in general nine points of inflexion; and we shall now shew that any straight line passing through two of them, must cut the curve again in a third. As before, we may assume for the general equation to curves of the third order, the form

$$pqr + ms^3 = 0,$$

since it contains the requisite number of independent constants. If with this equation we combine the equation $p = 0$, we get $s^3 = 0$; which shews that the three points in which the line

$p=0$ cuts the curve, are coincident; consequently the line $p=0$ has a contact of the second order with the curve at the point in which it intersects the line $s=0$, and that point is a point of inflexion. Similarly, each of the lines $q=0$, $r=0$ has a contact of the second order with the curve at the point in which it intersects $s=0$; and we thus obtain two other points of inflexion, lying in the same straight line with the first. Since out of nine things taken three at a time, twelve combinations may be formed such that no two of them have more than one element in common, it follows that the general equation of the 3rd degree may in twelve ways be put into the form $pqr+ms^3=0$; but in only one of them will the linear functions p, q, &c. be real, as only three of the points of inflexion can be real and six of them imaginary.

269. When a curve of the third order with three real asymptotes, has a double point, it will fall without, upon, or within the ellipse which touches in their middle points the three sides of the triangle formed by the asymptotes, according as it is a proper double point, a cusp, or a conjugate point.

If $u=0$ be the equation to a curve, then

$$d_{(x)}u + d_{(y)}u \cdot d_x y = 0;$$

and since at a double point $d_x y = \frac{0}{0}$, at such a point we must have $d_{(x)}u=0$, $d_{(y)}u=0$ (1); and to get the two values of $d_x y$ at the double point, we must have recourse to the second derived equation which, in consequence of the conditions (1), becomes

$$d^2_{(x)}u + 2d_{(x)}d_{(y)}u \cdot d_x y + d^2_{(y)}u \cdot (d_x y)^2 = 0; \qquad (2),$$

and according as this equation gives real, equal, or imaginary values of $d_x y$, i.e. according as

$$(d_{(x)}d_{(y)}u)^2 - d^2_{(x)}u \cdot d^2_{(y)}u >, =, \text{ or } < 0 \qquad (3),$$

there will be a proper double point, a cusp, or a conjugate point.

Now, taking the co-ordinate axes parallel to two asymptotes, let

$$(nx+a)(y+b)(c-nx-y) + m(y+gx+h) = 0,$$

or $u = pq(\beta - p - q) + ms = 0$ be the equation to a curve of the third order with three real asymptotes, where we have put $a + b + c = \beta$; then

$$d^2_{(x)}u = -2n^2q,\ d^2_{(y)}u = -2p,\ d_{(x)}d_{(y)}u = n(r - p - q);$$

therefore the condition (3) becomes

$$(r - p - q)^2 - 4pq = p^2 + q^2 + r^2 - \tfrac{1}{2}\beta^2 >, =, \text{ or } < 0;$$

which shews that a proper double point must fall without, and a conjugate point within, that ellipse which is the locus of the possible cusps; and which (Art. 263) touches in their middle points the three sides of the triangle formed by the asymptotes.

270. But if a series of curves of the third order having the same asymptotes, have each a double point such that one of the tangents passing through it is in a constant direction, then $d_x y$ will have a constant value κ suppose, which must satisfy equation (2). Hence substituting for $d_x y$, and for the differential coefficients of u their respective values, we get for the locus of the double points the equation

$$\kappa^2 q + \kappa(2p + 2q - \beta) + p = 0$$

representing that diameter to whose chords the above-mentioned tangent is always parallel.

271. If a curve of the fourth order have three proper double points, the six tangents at those points all touch a conic section.

Since the curve has three double points, if $p = 0$, $q = 0$, $r = 0$, be the equations to three straight lines passing through every two of them, and in the equation to the curve we suppose p to vanish, q^2r^2 must appear as a factor; similarly when q and r are supposed to vanish, p^2r^2 and p^2q^2 must appear respectively as factors; therefore the equation, since it is of the fourth degree, must be of the form

$$cp^2q^2 + bp^2r^2 + aq^2r^2 - 2pqr(a'p + b'q + c'r) = 0 \quad (1).$$

Now $q = \lambda r$ represents any straight line passing through the double point $q = 0$, $r = 0$; and if we combine it with the equation to the curve and reject the factor r^2 we obtain for result

$$r^2a\lambda^2 - 2pr(b'\lambda^2 + c'\lambda) + p^2(c\lambda^2 - 2a'\lambda + b) = 0;$$

and if λ be such that $c\lambda^2 - 2a'\lambda + b = 0$, then r will divide out, and the line $q = \lambda r$ in the positions determined by the two values of λ will be a tangent to the two branches that intersect in the double point $q = 0$, $r = 0$. Similarly $r = \mu p$ will become a tangent to each of the branches that intersect in the double point $p = 0$, $r = 0$, provided μ be taken such that $a\mu^2 - 2b'\mu + c = 0$; and $p = \nu q$ will touch each of the branches that intersect in the double point $p = 0$, $q = 0$, provided ν satisfies the equation $b\nu^2 - 2c'\nu + a = 0$. And it remains to shew that the six lines determined by these conditions are all tangents to the same conic section.

Now the condition that the straight line $lx + my + 1 = 0$ may be a tangent to the given conic section

$$Ay^2 + 2Bxy + Cx^2 + 2Dy + 2Ex + 1 = 0,$$

by eliminating x between these equations and expressing that the two values of y in the resulting equations are equal to one another, is found to be

$$(A - D^2)\,l^2 + (C - E^2)\,m^2 + 2\,(DE - B)\,lm + 2\,(BD - AE)\,l \\ + 2\,(BE - CD)\,m + AC - B^2 = 0.$$

If therefore, instead of $lx + my + 1 = 0$, we had $lp + mq + nr = 0$, where $p = 0$, $q = 0$, $r = 0$ are the equations to three given straight lines, the condition for this being a tangent to the conic section would lead to an equation of the form

$$al^2 + bm^2 + cn^2 + 2a'mn + 2b'ln + 2c'lm = 0,$$

a, b, c, &c. depending only on the constants in the given linear functions p, q, r, and in the given equation to the conic section.

Hence making successively l, m, n, $= 0$, we get for $mq + nr = 0$, $lp + nr = 0$, $lp + mq = 0$, being tangents to the same conic section, the conditions respectively $bm^2 + cn^2 + 2a'mn = 0$, $al^2 + cn^2 + 2b'ln = 0$, $al^2 + bm^2 + 2c'lm = 0$, which are the same conditions as we have already shewn to be fulfilled by the six tangents at the three double points of a curve of the fourth order; these six tangents consequently all touch the same conic section.

272. If a curve of the fourth order have three cusps of the first sort, the tangents at those cusps will intersect in a point.

If in equation (1) to curves of the fourth order in the preceding article we change the constants so that it becomes

$$p^2q^2 + m^2p^2r^2 + n^2q^2r^2 + 2pqr\,(mp + nq - mnr) = 0,$$

then the three proper double points are replaced by cusps of the first sort, because the two tangents at such double points become coincident; and the equations to the three tangents at these cusps are

$$mp - nq = 0, \quad q + mr = 0, \quad p + nr = 0,$$

which evidently represent three lines intersecting in a point.

273. If four double tangents be applied to a curve of the fourth order, the eight points of contact will be situated in one and the same conic section.

A double tangent is one which touches a curve in two distinct points; and cannot exist for curves of a lower order than the fourth, as the curve must be capable of being cut by a straight line in four or more points.

Suppose $u = 0$ to be the general equation of the second degree containing five constants; then the general equation of the fourth order may be put under the form

$$pqrs + mu^2 = 0, \qquad (1)$$

since this latter equation is of the 4th degree, and contains the requisite number, 14, of independent constants. If with equation (1) we combine the equation $p = 0$, we get $u^2 = 0$; therefore the four points of intersection of the straight line $p = 0$, with the curve (1), become coincident in pairs, and also coincide with the points in which the line $p = 0$, cuts the conic section $u = 0$; and as the same thing is true for each of the straight lines $q = 0$, $r = 0$, $s = 0$, it follows that the points of contact of the four double tangents will all be situated in the conic section $u = 0$. If therefore we have given a curve of the fourth order, and three of its double tangents P, Q, and

R; and if we know both pairs of the points of contact in P and Q, and one point of contact in R, and describe a conic section through those five points, it will cut R in the second point of contact, and it will likewise cut the curve itself in two additional points which are the points of contact of a fourth double tangent. For an example of this sort of curve, see p. 200.

274. To determine the double tangents of a curve of the fourth order.

Let $f(x, y) = 0$ be the general equation to curves of the fourth order, and $y = mx + n$ the equation to a double tangent; then eliminating y between these equations, there results for determining the abscissæ of the points of intersection of the line and curve, an equation of the form

$$ax^4 + bx^3 + cx^2 + dx + e = 0,$$

$$\text{or}\quad (x^2 + fx + g)\,(ax^2 + tx + u) = 0 \text{ suppose,}$$

the coefficients involving m and n to the fourth power.

Now this equation, since $y = mx + n$ represents a double tangent, has two pairs of equal roots, and must therefore have one of its quadratic factors $x^2 + fx + g$ suppose, in common with the derived equation $4ax^3 + 3bx^2 + 2cx + d = 0$; this latter equation must consequently be of the form

$$(x^2 + fx + g)\,(4ax + v) = 0.$$

Hence we have the identical equation

$$(ax^4 + bx^3 + cx^2 + dx + e)\,(4ax + v)$$
$$= (4ax^3 + 3bx^2 + 2cx + d)\,(ax^2 + tx + u);$$

and equating coefficients we get

$$a\,(v - 4t + b) = 0, \qquad bv - 3bt - 4au + 2ac = 0,$$
$$cv - 2ct - 3bu + 3ad = 0,$$
$$dv - dt - 2cu + 4ae = 0, \qquad ev = du;$$

substituting the values of t and u given by the first and last of these equations in the other three, we get three values of $v \div d$, which equated two and two give

$$(bd - 16ae)(6ad - bc) = (8ac - 3b^2)(cd - 6be),$$

$$(bd - 16ae)(6be - cd) = (8ce - 3d^2)(bc - 6ad);$$

and these two equations contain only the unknown quantities m and n, and serve to determine them.

275. A straight line will in general intersect a curve of the n^{th} order in n points; when it intersects the curve in fewer than n points, this happens because some of the points of intersection are situated at an infinite distance. The two constants that fix the position of a straight line may in general be determined so that two of its points of intersection with a given curve of the n^{th} order may be infinitely distant, in which case the line is called an asymptote, and may be considered as a tangent whose point of contact is at an infinite distance. But it is only for curves of a particular sort that more than two points of intersection of a straight line with them can be defective; when three points of intersection are wanting, the asymptote may be regarded as the tangent at a point of inflexion infinitely distant; when r points of intersection are wanting, the asymptote has a contact of the r^{th} order with the curve at an infinite distance, or is an osculating asymptote of that order.

276. A curve of the n^{th} order has in general n rectilinear asymptotes by which it is cut in $n(n-2)$ points lying in a curve of the $(n-2)^{th}$ order.

Let $u_n = 0$ denote the general equation to curves of the n^{th} order, and let p, q, &c. t be n linear functions of x and y, each containing two constants, and $u_{n-2} = 0$ the general equation of the $(n-2)^{th}$ order; then $u_n = 0$ may be put under the form

$$pqr \dots st + mu_{n-2} = 0;$$

for this latter equation is of the n^{th} degree, and contains the proper number $2n + 1 + \frac{1}{2}(n-2)(n+1) = \frac{1}{2}n(n+3)$ of independent constants. If now we combine the equation $p = 0$ with that to the curve, we get the equation $u_{n-2} = 0$ for determining the points of intersection of the line and curve; therefore two of those points are infinitely distant, or $p = 0$

is an asymptote; and the $n-2$ actual points of intersection lie in the curve $u_{n-2}=0$; the same is true for each of the other $n-1$ lines, $q=0$, $r=0$, &c.; therefore the curve will in general have n asymptotes, by which it is intersected in $n(n-2)$ points lying in a curve of the $(n-2)^{\text{th}}$ order.

277. A curve of the n^{th} order may in general, according as n is even or odd, be completely determined by means of $\frac{1}{2}n$ given points in it, and $\frac{1}{4}n(n+2)$ straight lines bearing fixed relations to it; or by $\frac{1}{2}(n-1)$ points, and $\frac{1}{4}(n+1)^2$ lines.

For if we denote $pqr \ldots st$, the product of n linear functions, by v_n, we have

$$u_n = v_n + m_2 u_{n-2};$$

similarly,

$$u_{n-2} = v_{n-2} + m_4 u_{n-4},$$

and so on, till, according as n is even or odd, we arrive at

$$u_2 = v_2 + m_n, \quad \text{or} \quad u_3 = v_3 + m_{n-1}v_1.$$

Consequently by successive substitutions, we get the general equation of the n^{th} order resolved into the form

$$v_n + \mu_2 v_{n-2} + \mu_4 v_{n-4} + \text{\&c.} = 0 \qquad (1),$$

the last term being μ_n or $\mu_{n-1}v_1$, according as n is even or odd.

Then the functions v_n, v_{n-2}, &c. are known, if the groups of straight lines which they respectively represent are given; and the number of these straight lines, which all bear a fixed relation to the curve, is evidently $\frac{1}{4}n(n+2)$ or $\frac{1}{4}(n+1)^2$, according as n is even or odd. Moreover if we know $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ points of the curve, according as n is even or odd, we may from (1) form the same number of linear equations among the coefficients μ_2, μ_4, &c., which will serve to determine them; so that every part of equation (1) will be known, and the curve completely determined.

According to the preceding notation the equation to a curve of the n^{th} order that has m osculating asymptotes of the r^{th} order will be

$$v_m u_{n-m} + \mu \cdot u_{n-r} = 0.$$

278. We are now able to explain the geometrical meaning of the several cases of curves of the third order noticed in Arts. 256 and 257; beginning with that of the cubic having three unequal roots.

When the three asymptotes are real, and no two of the lines $p = 0$, $q = 0$, $r = 0$, $s = 0$ parallel, the curve consists of three pairs of infinite branches, each pair turning their convexity towards, and tending to become coincident with, one of the asymptotes that lies between them. When the line $s = 0$ is parallel to one of the asymptotes $p = 0$, so that the equation to the curve becomes

$$pqr + m(p + k) = 0,$$

then $p = 0$ represents an osculating asymptote.

If the line $s = 0$ be removed to an infinite distance, so that the equation becomes $pqr + m = 0$, then all its asymptotes osculate the curve.

When two asymptotes are imaginary, the equation becomes $p(u^2 + v^2) + ms = 0$; in which case $p = 0$ represents an ordinary rectilinear asymptote, and the two imaginary asymptotes intersect in a real point. But if the line $s = 0$ be parallel to the real asymptote (so that the latter becomes an osculating asymptote), or be removed to an infinite distance, the equation is changed, respectively, into

$$p(u^2 + v^2) + m(p + k) = 0, \text{ or } p(u^2 + v^2) + m = 0.$$

279. When the cubic in Art. 256 has two equal roots, and the curve actually admits of two parallel asymptotes, (its equation being capable of being put into the form $pqr + ms = 0$, in an infinite number of ways,) as the number of constants is reduced to seven, the equation may be written either

$$p(q^2 - h^2) + m(q + k) = 0, \text{ or } p(q^2 + h^2) + m(q + k) = 0,$$

according as the two parallel asymptotes are real or imaginary; but if the line $q + k = 0$ be removed to an infinite distance so that the equation becomes $p(q^2 \pm h^2) + m = 0$, then $p = 0$ is an osculating asymptote. When $h = 0$, the parallel asymptotes become coincident.

When the two parallel asymptotes do not exist at any finite distance, the transformation into $pqr + ms = 0$ is impossible; but it can then be effected, in one determined way, into the form $p(q^2 + lp) + ms = 0$, where we see that $p = 0$ represents an ordinary rectilinear asymptote, and the parabola $q^2 + lp = 0$ is a curvilinear asymptote, since instead of six it has but two points of intersection with the curve; and the points of intersection both of the rectilinear and parabolic asymptotes with the curve lie in the line $s = 0$. If preserving its due number of constants the equation be put under the form

$$(p + h)(q^2 + lp) + m(q + k) = 0,$$

we see that the parabola $q^2 + lp = 0$ meets the curve only in one point, and is consequently an osculating asymptote of the fifth order; and if the line $q + k = 0$ move off to an infinite distance so that the equation becomes

$$(p + h)(q^2 + lp) + m = 0,$$

the parabola $q^2 + lp = 0$ then becomes an osculating asymptote of the sixth order.

280. When the cubic in Art. 256 has three equal roots, the equation to curves of the third order may in a particular case be reduced to the form $p^3 + mq = 0$ representing the cubical parabola, the nature and figure of which are well known; and generally to the form $p^3 + lq^2 + ms = 0$, which only admits the semi-cubical parabola $p^3 + lq^2 = 0$ for asymptote, but has no rectilinear or parabolic asymptote; moreover if $s = p + h$, or becomes constant, the asymptote osculates the curve. When the number of constants is reduced to six, the equation may be transformed, in one determined way, into $p(p^2 + lq) + m = 0$, which represents a curve called by Newton the Trident, having for asymptotes both the parabola $p^2+lq=0$, and the line $p = 0$ which is a diameter of the parabola, by neither of which is it intersected; it has an infinite branch on each side of $p = 0$, and two others, one on the outside of the first, and the other on the inside of the second branch of the parabolic asymptote.

= 1.

x

10

x

x'

x

24

2.

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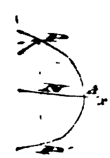

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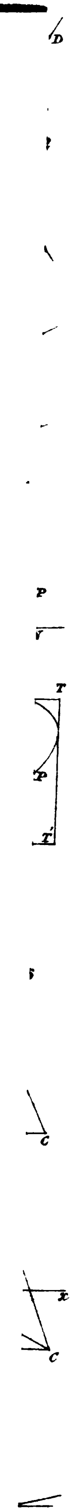
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